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ATMOSPHERIC-PRESSURE GAS LASERS

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FINAL TECHNICAL REPORT ON ATMOSPHERIC PRESSURE LASERS

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submitted by

Hermann A. Haus
Professor of Electrical Engineering

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I. Introduction

The work for this research contract on atmospheric-pressure gas lasers had as its main objectives the generation of short laser pulses through mode locking and cavity dumping, the study of the nonlinear amplification processes in a transversely excited atmospheric pressure laser, and development of computer programs to check the observed amplifier responses against models of the relaxation processes of the CO_2 molecule in a laser discharge.

Thus, the proposed investigation concentrated on three methods by which pulses could be made both more energetic and of shorter duration:

(a) mode locking

(b) cavity dumping

(c) nonlinear amplification.

The work on nonlinear processes in CO_2 systems was originally begun in order to demonstrate the possibility of amplifying pulses of duration short compared with the inverse bandwidth of the amplifying medium, without the undesirable pulse-broadening associated with <u>linear</u> amplification. Obviously, this can be done only if the effective bandwidth of the nonlinear amplification is broader than that of linear amplification. Line broadening due to optical saturation by cw radiation is well known.[1] The purpose was to demonstrate that, dynamically, line broadening via saturation could be utilized to amplify over a broader bandwidth than that accorded to linear amplification. It was also hoped that dynamic pulse shaping could be utilized to lead to additional pulse shortening.[2] This dynamic pulse shaping is closely related to the optical "nutation" effect.[3] The principle of pulse shaping and shortening by optical nutation was demonstrated[4] just before the initiation of the present contract.

The work under this contract was concerned with the improvement of the pulse shaping and shortening, and with the effort to increase energy extraction in the nonlinear amplification. The modelocking-cavity dumping scheme to produce single, high power pulses was abandoned in favor of a pulse selection outside the cavity. It turned out that the ultimate power achievable by the pulse selection outside the cavity was higher than the one achieved through cavity dumping. The electrooptic crystal used for cavity dumping depressed the peak power of the laser more than the loss of power due to the partial mirror transmission. With a conventional TEA laser, pulses of 500 kW peak power and 2 ns long were obtained. These were used as the experimental probe for short pulse amplification studies.

These studies were aimed at determining the limits on energy extraction and achievable pulse-widths in the pulse generation and nonlinear amplification. One problem encountered in the amplification of short pulses in CO_2 is the incomplete energy extraction from the rotational levels if the pulse duration becomes comparable to the rotational relaxation time.

In fact the extraction time of the energy from all rotational levels is roughly[5] $N_{eff}\,\tau_{rot}$, where τ_{rot} is the rotational relaxation time and N_{eff} is the effective number of rotational levels (a function of temperature, a number of the order of 10). One of our studies concentrated on the utilization of the energy stored in the vibrational levels as discussed in Section I.

In the analysis of the data of Section I, we used a computer program that treated the rotational-vibrational relaxation of CO_2 under short-pulse excitation. This computer program was developed by A. Ross of this laboratory who was supported by a NSF contract. The computer program used curves obtained by Nighan[6] for the pumping rates of the energy levels by the discharge. Nighan's work is based on a full treatment of the Boltzmann equation and the evaluation of the pumping rates requires an elaborate computer program and considerable computer time. We considered it expedient to develop an analytic theory of electronic pumping rates using approximate cross sections for the vibrational and electronic excitation rates. The advantage of this approach is that it lends insight into the important mechanisms determining laser efficiency. A description of the theory forms section II, details appear in the literature.[7]

The study of energy extraction by short pulses showed the importance of obtaining short and energetic pulses by the mode-locking process in the first place. Saturable absorber modelocking has, thus far, produced much shorter modelocked pulses than forced modelocking in those systems in which both methods have been employed successfully. Therefore we looked seriously at modelocking of the CO_2 system by saturable absorbers, both experimentally and theoretically. We found that no adequate analytic theory of saturable absorber modelocking has been published in the literature. Therefore, we set out to develop such a theory with the aim of attaining a description of saturable absorber modelocking as complete as that presented by Kuizenga and Siegman [8] for forced modelocking. We were successful in doing this. We believe that the theoretical work on saturable absorber modelocking is the most important success of the work performed under this contract. Three papers on the subject of modelocking have appeared, or will appear shortly. [9,10,11] The last of these was carried out at Bell Laboratories by the principal investigator, who spent a year with the group working on modelocking of dye lasers (Prof. Szöke took his place during his absence). A further paper has been submitted for publication. It forms Appendix I. We shall not reproduce the papers here but give, in Section III, an account of the salient features of the theory and its success in explaining certain experimental observations on passively modelocked systems.

The experimental work on saturable absorber modelocking concentrated on the CO_2 system using SF_6 as a saturable absorber. In order to describe the action of the SF_6 absorber adequately, a model of SF_6 was developed which accounts better for the transient absorption of SF_6 as a function of pulse energy and pulsewidth than previously proposed models. [12,13] This topic is taken up in Section IV.

I. Energy Extraction from Non-Lasing Vibrational Levels

As mentioned in the introduction, we found it expedient to select modelocked pulses outside the cavity, rather than via cavity dumping, because the final energy of the pulse was higher with this method of pulse selection. Also, in this way more than one pulse could be "clipped" from the pulse train. Two pulses in succession were used to probe the vibrational population recovery in a CO_2 TEA amplifier and hence the possibility of additional energy extraction from the amplifier, within times corresponding to the vibrational relaxation times.

The amplifier was filled with a mixture of CO_2, N_2 and helium at 200 Torr and the partial pressure of CO_2 was varied in the experiment from 5 Torr to 40 Torr. The individual input pulse lengths were 2 ns, the spacing between the pulses, 12 ns. The first laser pulse was made to enter the amplifier $\sim$30 μs after the application of the discharge current pulse. The recovery of the gain of the second pulse as a function of partial pressure of CO_2 was determined. The total pressure in the amplifier was sufficient for appreciable equilibration of the rotational population distribution within the duration of one pulse. Furthermore, since the rotational relaxation is mainly a function of the total pressure, and not of the partial pressure of CO_2, variation of the partial pressure did not affect the recovery time of the rotational population distribution. Therefore the increased recovery as a function of increasing pressure is attributable to V-V relaxation processes. The level which has an appreciable population after equilibration of the vibrational temperatures and which can feed the upper laser level appreciably within 12 ns at the partial pressure used is the 011 level. Hence this experiment ascertains the rate of relaxation of the 011 level and the degree of population recovery by means of this relaxation process. Figure 1 shows the result of the experiment. At the very low partial pressures, when the relaxation mechanism is not operative within 12 ns, the gain of the second pulse compared with that of the first one has been reduced by 20% because of the population depletion of the first pulse. The experimentally observed recovery as a function of partial pressure is shown. Two theoretical curves computed on the assumption that the recovery is due entirely to the feeding of the upper laser level by the 011 level population are shown as dashed curves. The two curves were computed for two values of population inversion changes consistent with the gain changes caused by the first pulse. This shows that the major portion of the recovery is explainable by this mechanism. The fact that the experimentally observed recovery is larger than the one predicted on the basis of the 011 level relaxation is attributable to the more effective relaxation of the lower laser level with increasing partial pressure, an effect not taken into account in the theory which assumed that even at the low partial pressures the lower laser level relaxes completely within the 12-ns interval.

Details of the experiment are given in the RLE Quarterly Progress Report No. 111, Oct. 15, 1973. The theoretical computations are contained in the Master's thesis of Y. Manichaikul.[14] The conclusion drawn from the experiment is that energy storages available in vibrational combination modes can be utilized if the pulse to be amplified is lengthened. The present experiment indicates the magnitude of the effect and the time scales at which such utilization occurs.

II. Electron Distribution and Lasing Efficiency

Computer codes evaluating the electron distribution and the pumping of laser levels have been developed in several laboratories, notably at United Aircraft Corporation.[15] Whereas good quantitative results can be obtained from these programs, it is difficult to cull from them physical insight about the influence of discharge parameters on gain and lasing efficiency, without carrying out costly computations. With this in mind, we have developed a simplified model for the electron distribution and the pumping of the electron laser levels which is amenable to closed-form solutions. The details of this work have been published.[7] Here we give a brief summary of the salient features.

The energy transfer of the electrons to the vibrational levels and to the electronic radiation levels (which are useless energy sinks in the case of a IR molecular laser) is controlled by the shape of the velocity distribution function of the discharge electrons. This distribution function is approximately spherically symmetric and hence describable in terms of a function $f_o(v)$ of a scalar v, the magnitude of the electron velocity.

The basic equation governing $f_o(v)$ is derived from the Boltzmann Equation:

$$4\pi \frac{d}{dv}\left[\frac{v^2}{3}\left(\frac{e}{m}\right)^2 \frac{E^2}{\nu_c}\frac{df_o}{dv}\right] = -\,4\pi v^2 \left(\frac{\partial f}{\partial t}\right)_{inel}, \qquad (1)$$

where ν_c is the frequency of momentum transfer collisions and E is the electric field in the discharge. The term on the left-hand side represents the rate of entry of electrons into the velocity range v, $v + dv$ produced by the driving electric field. The rate of entry of electrons is equal to the rate of exit caused by the inelastic collisions, the term on the right-hand side of the equation. The distribution function $f_o(v)$ is normalized:

$$\int 4\pi v^2 \, f_o(v) \, dv = 1.$$

An analytic treatment results from an appropriate modeling of the inelastic collision term on the right hand side of (1). One approximates the collision cross sections of the electrons with the molecules, which exhibit resonant peaks as functions of energy, by impulse functions of energy. The main collision processes are the vibrational energy excitations with peaks near 2 eV (for N_2 and CO_2), and electronic excitations, with peaks that may all be lumped into a single excitation cross section, as long as the average electron energies are as low as those generally produced in molecular-laser gas discharges.

With these assumptions, closed form solutions can be obtained for $f_o(v)$ (or $f_o(u)$ where $u = mv^2/2c$).

Some interesting conclusions were obtained from the analysis.

(a) In an E-beam excited discharge we find that the V-I characteristic of the sustainer electrodes remains unchanged by the presence, or absence, of lasing. For a given power input per unit volume, lasing redistributes the flow of energy among the levels, changes the electron distribution, but does not affect the conductivity of the medium. This conclusion turned out to be rather model independent. The sole source for this somewhat surprising result is the assumption that the elastic collision frequency is energy independent.

(b) The efficiency of a E-beam laser peaks at a value of electric field that maintains the average energy of the electrons well below 2 eV, the energy at which the vibrational excitation cross section was assumed to peak. The reason for the rapid decrease of efficiency with increasing electric field beyond this critical field is that the energetic "tail" of the electron distribution excites the electronic levels excessively.

While these conclusions could have been culled from Nighan's computer analysis, it is helpful that the analytic treatment reaches these conclusions in a transparent way and, in fact, gives results even in (approximate) quantitative agreement with Nighan's results.

III. Theory of Passive Modelocking of a Homogeneously Broadened Laser

The purpose of the new theory of passive modelocking[9,10,11] is to provide an analytic description of the phenomenon equivalent to the one presented by Siegman and Kuizenga[8] for the case of active modelocking. Passive modelocking is a phenomenon more complicated than active (forced) modelocking because the modulation provided by the saturable absorber is a function of the pulse amplitude and shape passing through the absorber, and depends on the relaxation time of the absorber.

Just as in the case of active modelocking, the theory of passive modelocking is concerned at first with cw operation, because it is simpler. We shall first outline the theory and then present arguments that many results of the theory are applicable to the transient buildup of modelocking.

The analysis makes the following assumptions and approximations:

(a) The laser medium is homogeneously broadened.

(b) A pulse passing through any one of the elements of the system, saturable absorber, gain medium, etc., is modified only slightly (say 20 percent gain or loss) on one single pass. This assumption enables one to expand exponentials to first order in their argument.

(c) The dispersion of the system as a function of frequency may be expanded to second order in frequency (this assumption is legitimate if the laser pulse spectrum is narrow compared with the bandwidth of the system).

Under these assumptions one arrives at the following equation for the envelope $v(t)$ of the electric field, $E(t) = v(t) \exp j\omega_o t$, of the pulse traveling in one direction between the mirrors of the the laser cavity:

$$\left[1 + jb - g\left(1 + \frac{1}{\omega_L{}^2}\frac{d^2}{dt^2}\right) + \frac{\delta + g}{\omega_L}\frac{d}{dt}m(t)\right] v(t) = 0 \qquad (1)$$

Here the factor 1 expresses the effect of the normalized cavity loss on the pulse, jb is the reactive effect caused by oscillation of the cavity modes off their natural resonance frequency; $g\left(1 + \frac{1}{\omega_L{}^2}\frac{d^2}{dt^2}\right)$ is the operator describing the effect of the dispersive laser medium gain on the pulse envelope; ω_L is the laser medium linewidth and g is the gain at line center per pass normalized to the loss per pass. The operator d^2/dt^2 is a diffusion operator in "time-space" which accounts for the spreading of the pulse in time because of medium dispersion. The operator $\frac{\delta + g}{\omega_L}\frac{d}{dt}$ describes the net time advance δT (or delay $-\delta T$ of the pulse) with respect to the empty cavity roundtrip time T_R of the cavity as caused by the laser medium dispersion, as well as that induced by the modelocking modulation $m(t)$ of the modelocking element. Equation (1) sets equal to zero the sum of the modifications of the pulse due to cavity loss, laser gain, pulse-advance and modulation, i.e. it states the requirement of a steady state. The fundamental relation (1) describes both the forced, and passive methods of modelocking. Indeed, if the modelocking is forced, then

$$m(t) = 2M[1 - \cos \omega_m t]$$

where M expresses the modulation amplitude normalized to the cavity loss, and ω_m is the frequency of modulation. When (2) is introduced into (1) and after the change of variable $v(t) = V(t) \exp (\delta + g/2) \omega_L t$, one obtains an equation that is identical in form to the equation of motion of quantum mechanics of a particle in a periodic potential (the time t must be replaced by the spatial coordinate x). The result of Siegman and Kuizenga is obtained through one further approximation. If the modulation is strong, leading to well isolated pulses, then $1 - \cos \omega_m t$ may be approximated by $1/2\ (\omega_m t)^2$ within the time interval occupied by the pulse. Under this approximation the well known gaussian pulse solution[8] is obtained. In addition to the gaussian, higher order Hermite gaussian solutions are obtained as well as previously pointed out by Haken and Pauthier.[14] We were able to show[9] that all pulse solutions, except the lowest order Hermite Gaussian, are unstable for synchronous modulation (ω_m is set equal to the $2\pi/T_P$ where T_P is the cavity roundtrip time including the additional delay caused by dispersion of the laser medium).

Hence, Equation (1) describes fully the forced-modelocking results of Siegman and Kuizenga. Saturable absorber modelocking is described, when the modulation $m(t)$ is determined self consistently with the pulse intensity passing through the saturable absorber. For an absorber with relaxation time fast compared to the pulse duration,

$$m(t) \propto \frac{1}{\left(1 + \dfrac{v^2}{P_A}\right)} \simeq 1 - \frac{|v|^2}{P_A}$$

where P_A expresses the saturation power of the absorber, if v is so normalized that $|v|^2$ is equal to the instantaneous power in the pulse. The resulting nonlinear differential equation can be solved. The solutions are elliptic integrals which reduce to a secant hyperbolic in the limit of well separated pulses.[10] If the absorber is slow compared with the pulse duration, then one faces a stability problem: if the absorber does not "shut-off" after passage of the pulse, perturbations following the pulse will grow and the solution is unstable. Stable pulses can be obtained if the laser gain is reduced by the passage of the pulse so as to shut off the net gain after the pulse has passed. Again, under certain, not too restrictive assumptions, the pulse solutions are secant hyperbolics.[11] It can be shown in general that the wings of pulses produced by saturable absorber modelocking must be exponential, not gaussian.[11] This is of importance when using these pulses in nonlinear time resolved spectroscopy on a time scale comparable to the pulse duration. This exponential time dependence of the pulse wings has been observed experimentally on modelocked dye laser pulses.[15]

So far we have discussed theories of cw passive modelocking We contend that the theories are relevant to the transient buildup of modelocking provided certain criteria are met. We further claim that these criteria must be met if the transient modelocking of a gain switched laser is to provide deterministic (nonstatistical) pulse trains, repeatable from "shot" to "shot". We shall come back to this point after a brief discussion of previous work on the transient buildup of modelocked pulses.

Much computational work was done in the literature[16-25] on the transient buildup of saturable absorber modelocking. The prevailing view was the one originally proposed by Letokhov[17]: The buildup of modelocked pulses is started by initial noise spikes of random width and height traveling between the two laser reflectors. From among these spikes, those of largest energy are selected by the "gating action" of the saturable absorber. A natural consequence of this model is that the predicted pulsed output of a Q-switched or gain switched laser is statistical in nature.

Experimentally, no doubt, the modelocked pulse from gain switched Nd Glass or Nd YAG do exhibit statistical behavior. This can be undesirable if predictable pulses are to be produced from shot to shot. Special experimental efforts have been made to produce repeatable pulse trains of modelocked pulses. This has always been found to necessitate operation of the laser near threshold. It is our contention that there is a limit of transient modelocked operation in which the modelocking is entirely deterministic. The steady state theory of modelocking bears on the buildup of these "deterministically" modelocked pulses.

In support of this picture we cite the following argument. Initially, the laser builds up from spontaneous emission noise. In order to reach power levels that cause saturation of the absorber, the power has to grow by a factor of 10^{15} or so. Such large growth leads to gain-bandwidth narrowing that is proportional to the logarithm of the growth factor--an appreciable factor. Hence the power traveling in one direction inside the cavity is <u>smoothed</u> by the bandwidth narrowing. All of this still occurs during the buildup process. Of all Fourier components--harmonics of $2\pi/T_R$ --the lowest Fourier component grows fastest initially. Hence the "smoothed" average (with some statistical structure) will evolve into a growing sinusoidal perturbation of a "dc background". If the system is not driven excessively above threshold, then the time evolution of the pulse may be treated adiabatically, as a succession of quasisteady states. This theory is outlined in Appendix II.

IV. Short Pulse Saturation of High Pressure SF_6

Interest in the short pulse saturation properties of SF_6 has been stimulated by its usefulness as a Q switching[26] and modelocking[27] element for the P branch of the 10.4 μ band of the CO_2 laser. More recently, work on laser isotope separation using SF_6 has intensified that interest.[28]

Our primary motivation in carrying out the present study of SF_6 short pulse saturation was to determine the characteristics of the medium appropriate to modelocking the high pressure CO_2 laser. The modelocking mechanism is the bleaching of the SF_6 absorption on the time scale of the modelocked pulses. Thus, the absorption is, in part, due to a set of levels having a relaxation time on the order of or less than the typical width of the modelocked pulse. In other words, the attenuation experienced by an intense pulse of duration τ_p in propagating through a cell of SF_6 should be less than that experienced by a quasi-continuous signal of the same energy. By quasi-continuous we mean a pulse of duration T_p greater than that of the modelocked pulse but less than the V-T relaxation time of SF_6.

We have completed a set of experiments which verify this hypothesis. By alternately irradiating a 1 mm long cell of SF_6 with the single mode output and the modelocked output of a high pressure TEA CO_2 laser we have measured the difference in transmission of the cell for equal values of the total energy in the TEA pulse. Single mode operation was achieved via a 2.5 cm long intracavity cell of low pressure SF_6 (< 1 torr). Modelocking was produced by an intracavity acousto-optic modulator tuned to the cavity mode spacing at approximately 42 MHz. The modelocked pulses were 2 ns long spaced 24 ns apart, thus giving approximately a 12:1 peak power enhancement over the single mode TEA pulse of equal energy.

The output of the TEA laser was passed through a 3 mm aperture to ensure a nearly uniform intensity distribution across the beam and focussed by a salt lens to a .2 mm diameter spot inside the SF_6 cell. A second lens recollimated the beam before it was detected by a Au:Ge detector. Different values of input intensity were achieved by placing calibrated CaF_2 attenuators in the path of the beam. From the detector the signal was amplified by an amplifier which was slow ($\sim$ 5μs) compared to the TEA pulse ($\sim$ 250 ns) so that its response depended only on the total energy in the TEA pulse. The amplifier output was then fed into a RIDL 400 channel pulse height analyzer which displayed the distribution of transmitted TEA pulse energy for successive shots of the laser. The use of the analyzer enabled us to distinguish between shot to shot fluctuations in the laser output energy ($\sim$ 10%) and differences in absorber transmission.

Results of the transmission measurements at various SF_6 pressures for the P(22) line of 10.6 μ CO_2 output are shown in Figure 2. The modelocked pulse transmission is always higher than that of the single mode pulse. Furthermore, we observe that the saturation curves appear to be roughly asymptotic to the small signal absorption values measured by Wood et al.[29] indicating that the absorption cross-section does not change appreciably with pressure.

The complexity of the energy level structure of SF_6 makes it difficult to analyze. Models developed by Burak et al.[12] and Brunet[30] show excellent agreement with experiments on the cw saturation of SF_6. The response of SF_6 to short pulses, however, is not well understood. Recent work by Oppenheim and Kaufman[13] shows that the pulse saturation behavior of SF_6 can be dramatically different from the cw behavior. However, the model adopted by Oppenheim and Kaufman is unrealistic in that it assumes an instantaneous relaxation rate for the excited state transition. Pulse transmission measurements have also been performed by Armstrong and Gaddy,[31] but their model ignores V-V equilibration and neglects the $\nu_3 \rightarrow 2\nu_3$ excited state absorption. The model's prediction of a $\nu_6 \rightarrow \nu_3 + \nu_6$ relaxation time which is much faster than gas kinetic and a V-T time which is three orders of magnitude faster than previously determined values[32] is a measure of its weakness.

We have developed a simple closed form theory to explain the saturation curves for both the single mode and modelocked pulses. Our approach is similar to that of Oppenheim and Kaufman in that we consider only the absorption of the ν_3 mode and neglect the coupling to other vibrational modes of the molecule. In contrast to previous work, however, we include the three fold degeneracy and full V-V equilibration of the ν_3 mode and the saturation of the excited state absorption $\nu_3 \to 2\nu_3$. To do so we adopt the model shown in Figure 3 in which the following notation is used:

N_1, N_2, N_2', N_3	Population densities in the three vibrational levels which are directly interacting with the incident field.
N_{11}, N_{22}, N_{33}	Population densities which do not interact directly with the field.
σ_o, σ_1	Absorption cross-sections of the ground state and first excited state respectively.
R	Rotational relaxation rate.
β_ν	Fraction of total number of molecules in the νth vibrational level which interact directly with the field.
k_{vv}	Rate constant for equilibration among the vibrational levels.

We have omitted any reference to the V-T relaxation time because it is slow compared to our pulse duration. Measured values of the V-T rate place it at 122 μs/torr[32] which is much greater than our 250 ns pulsewidth even at 250 torr, the maximum pressure used in our experiments.

The theoretical saturation curves computed from the above model are compared to the experimental single mode P(22) transmission data in Figure 4. Good agreement is found at low pressures. At higher pressures, however, SF_6 is harder to saturate than the model predicts. This discrepancy can be attributed to the coupling between ν_3 and other vibrational modes of the molecule which is neglected in our model. The difference between theory and experiment indicates that approximately three vibrational modes contribute to the absorption at pressures of 100 torr and greater.

The details of this work will be submitted for publication.

Conclusions

Of the results obtained by the research summarized here the most important one is believed to be the development of a theory of saturable absorber modelocking. In the process of its application to CO_2 laser modelocking a better understanding of SF_6 as a modelocking absorber has been obtained. Numerous publications have resulted from the work, some bearing specific acknowledgement to this contract, some which have been conceived as an outgrowth of this work while the principal investigator was on a leave of absence at the Bell Laboratories, in part with the objective of applying the new theory of saturable absorber modelocking to the cw modelocked dye laser producing subpico-second pulses at the Bell Laboratories. A list of publications, which grew out of this work, published, accepted, and/or submitted is summarized below:

With Contract Acknowledgment Published:

[i] W. P. Allis, H. A. Haus, "Electron Distributions in Gas Lasers," J. Appl. Phys., 45, 781-791 (1974).

[ii] H. A. Haus, "A Theory of Forced Modelocking," IEEE J. Quantum Electronics, QE-11, 323-330 (1975)

[iii] H. A. Haus, "Theory of Modelocking with a Fast Saturable Absorber," J. Appl. Phys., 40, 3049-3059 (1975)

Submitted:

[iv] P. Hagelstein, C. P. Ausschnitt, "Shape and Stability Dependence of Passively Modelocked Pulses on Absorber Relaxation Time," to J. Appl. Phys.

Without specific contract acknowledgment, because carried out at Bell, but conceived as an outgrowth of the present work:

To be published:

[v] H. A. Haus, "Theory of Modelocking with Slow Saturable Absorber," to be published in J. Quantum Electronics, September (1975)

[vi] H. A. Haus, C. V. Shank, E. P. Ippen, "Shape of Passively Modelocked Laser Pulses," Optics Comm.

[vii] H. A. Haus, "Model of cw Modelocked Dye Laser," to J. Appl. Phys.

[viii] H. A. Haus, "Parameter Ranges for cw Passive Modelocking," to IEEE J. Quantum Electronics.

Personnel Working on Contract

Students:

Yongyut Manichikul (Master's Degree, Thesis title: "Generation and Amplification of High Intensity Nanosecond TEA CO_2 Laser Pulses," September 1973).

Peter Hagelstein (Bachelor's Degree, Thesis title: "Modelocking by Saturable Absorber with Nonnegligible Relaxation Time," to be submitted in September 1975).

Christopher Ausschnitt (Doctor's Degree, Thesis title: "Mode-locking Studies on High Pressure CO_2 Lasers," to be submitted in September 1975).

Faculty:

H. A. Haus

A. Szöke (took Professor Haus' place during leave of absence at Bell Laboratories)

W. Allis*

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* Not supported financially.

References

[1] A. A. Vuylsteke, Elements of Maser Theory, Van Nostrand Princeton (1960)

[2] W. G. Wagner, H. A. Haus, K. T. Gustafson, IEEE J. Quantum Electron., QE-4, 267 (1968)

[3] C. Hocker and C. Tang, Phys. Rev. Lett., 21, 591 (1968)

[4] P. W. Hoff, H. A. Haus, T. J. Bridges, Phys. Rev. Lett. 25, 88 (1970)

[5] J. F. Figuera, W. H. Reichert, G. T. Schappert, T. F. Stratton, C. A. Fenstermacher, Appl. Phys. Lett., 22, 216 (1973)

[6] W. L. Nighan, Phys. Rev., A2, 1989 (1970)

[7] W. P. Allis, H. A. Haus, J. Appl. Phys., 45, 781 (1974)

[8] D. J. Kuizenga, A. E. Siegman, IEEE J. Quantum Electron., QE-6, 694 (1970)

[9] H. A. Haus, IEEE J. Quantum Electron., QE-11, 323 (1975)

[10] H. A. Haus, J. Appl. Phys., 40, 3049 (1975)

[11] H. A. Haus, to be published in IEEE J. Quantum Electron., Sept. (1975)

[12] I. Burak, J. I. Steinfeld, D. G. Sutton, J. Quantum Spectrosc. Radiat. Transfer., 9, 959 (1969)

[13] U. P. Oppenheim, Y. J. Kaufman, IEEE J. Quantum Electron., QE-10, 533 (1974)

[14] H. Haken, M. Pauthier, IEEE J. Quantum Electron., QE-4, 454 (1968)

[15] H. A. Haus, C. V. Shank, E. P. Ippen, paper accepted by Optics Comm.

[16] V. S. Letokhov, V. N. Marzov, Sov. Phys. JETP, 25, 862 (1967)

[17] V. S. Letokhov, Sov. Phys. JETP, 27, 746 (1968)

[18] V. S. Letokhov, Sov. Phys. JETP, 28, 562 (1969)

[19] V. S. Letokhov, Sov. Phys. JETP, 28, 1026 (1969)

[20] V. S. Letokhov, Sov. Phys. JETP, 28,

[21] J. A. Fleck, Appl. Phys. Lett., 12, 178 (1968)

[22] J. A. Fleck, J. Appl. Phys., 39, 3318 (1968)

[23] J. A. Fleck, Phys. Rev. Lett., 21, 131 (1968)

[24] J. A. Fleck, Phys. Rev., B1, 84 (1970)

[25] E. Mathieu and H. A. Weber, Z. Angew, Math. Phys., 22, 458 (1971)

[26] O. R. Wood, S. E. Schwarz, Appl. Phys. Lett., 11, 88-89, Aug. (1967)

[27] A. Nurmikko, T. A. DeTemple, S. E. Schwarz, Appl. Phys. Lett., 18, 130, Feb. (1971)

[28] J. L. Lyman, R. J. Jensen, J. Rink, C. P. Robinson, S. D. Rockwood, Appl. Phys. Lett., 27, 87, July (1975)

[29] O. R. Wood, P. L. Gordon, S. E. Schwarz, IEEE J. Quantum Electron., QE-5, 502, Oct. (1969)

[30] H. Brunet, IEEE J. Quantum Electron., QE-6, 628, Nov. (1970)

[31] J. J. Armstrong, O. L. Gaddy, IEEE J. Quantum Electron., QE-8, 797, Oct. (1972)

[32] J. I. Steinfeld, I. Burak, D. G. Sutton, A. V. Nowak, *J. Chem. Phys.*, **52**, 5421, May (1970)

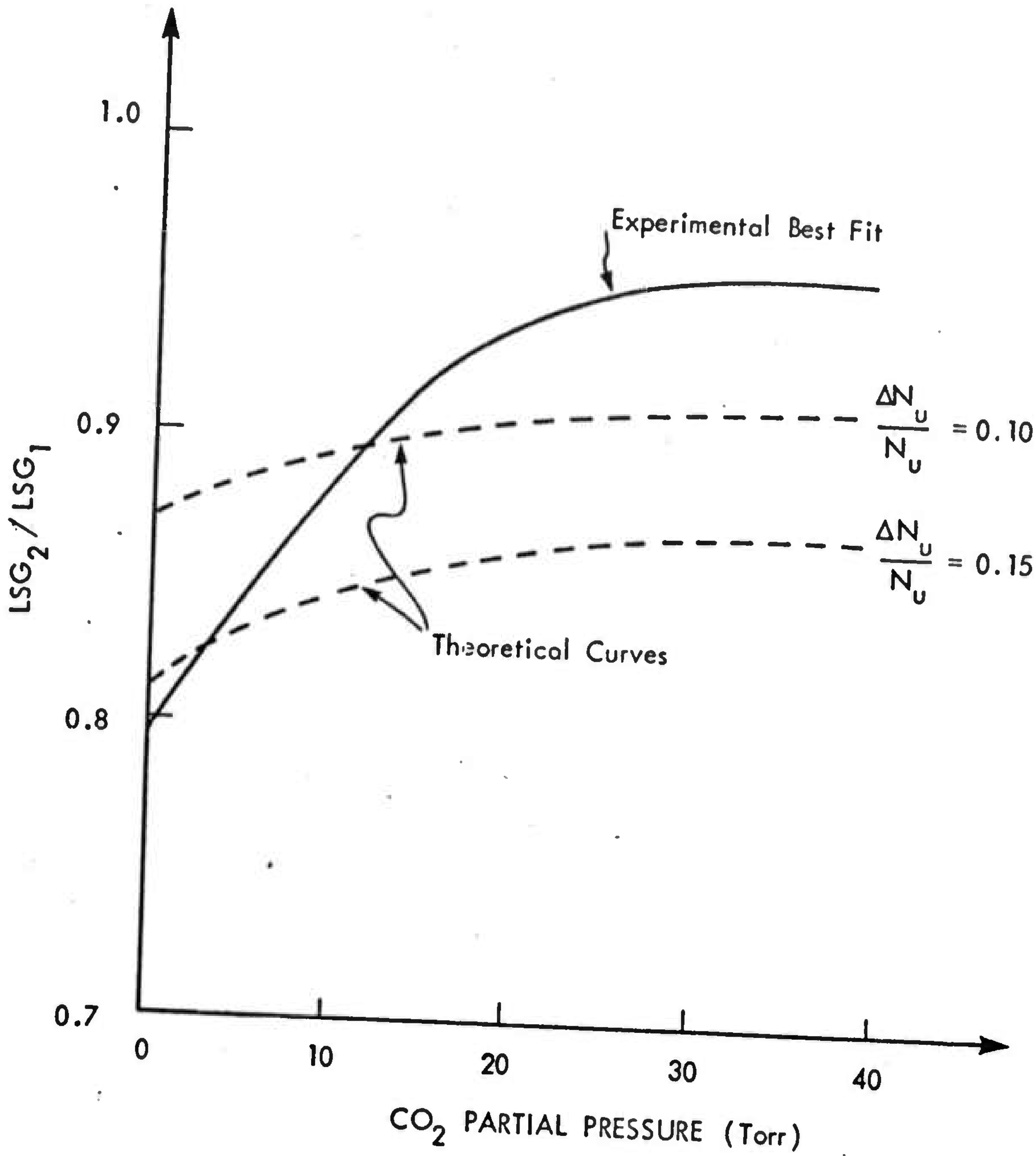

Figure 1. Theoretical plot of LSG_2/LSG_1 against CO_2 partial pressure where LSG_i is the large signal gain experienced by the i-th pulse.

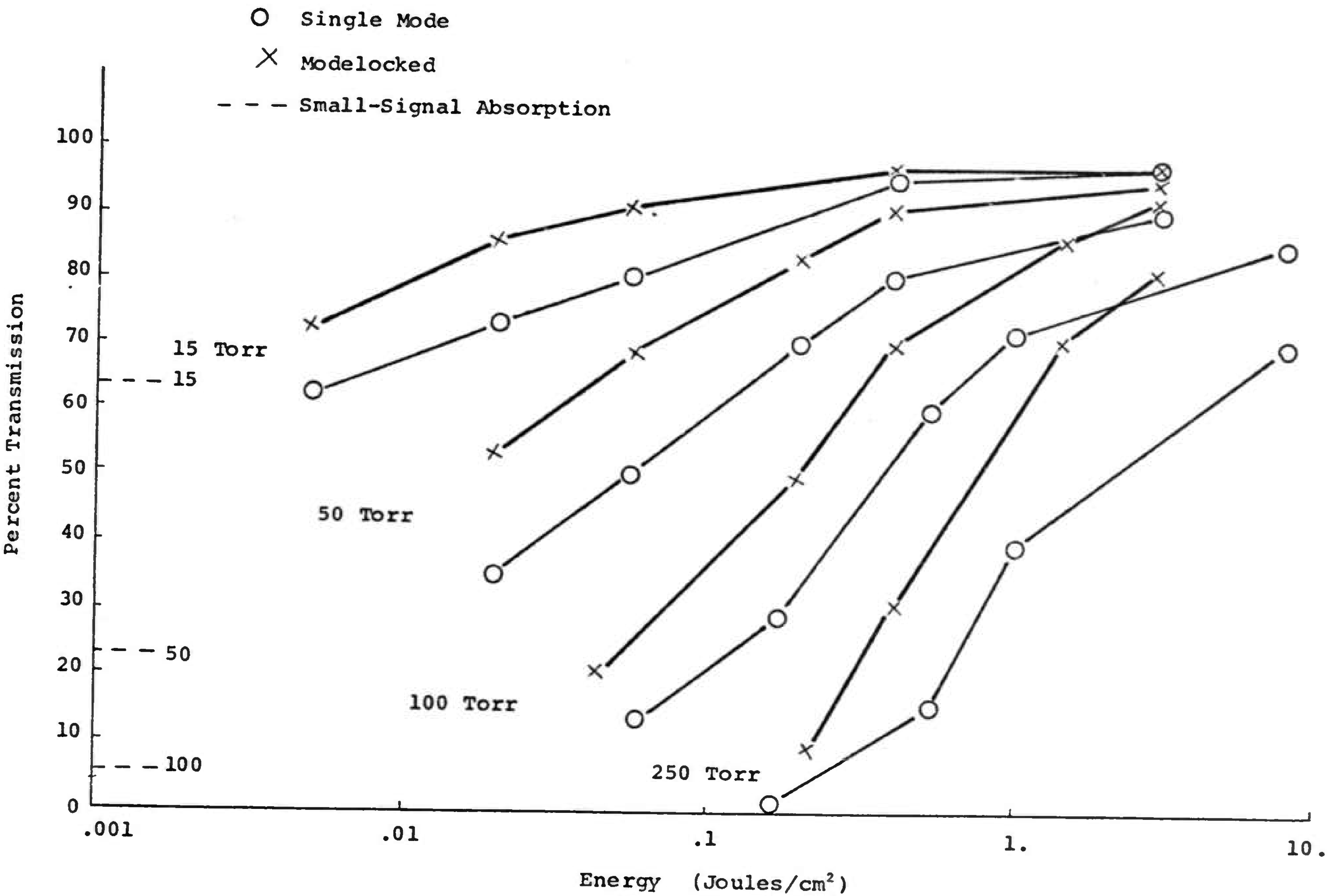

Figure 2. SF_6 Transmission on the P(22) Line

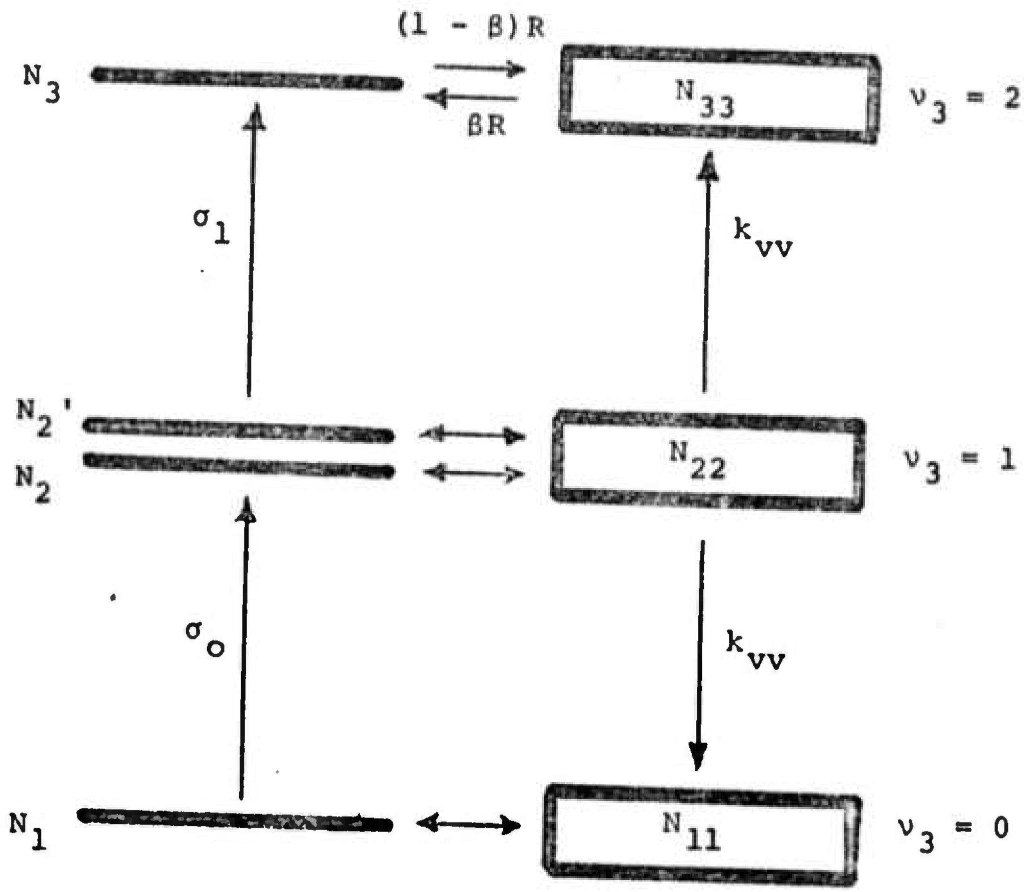

Figure 3. Energy level diagram for absorption by the ν_3 mode of SF_6

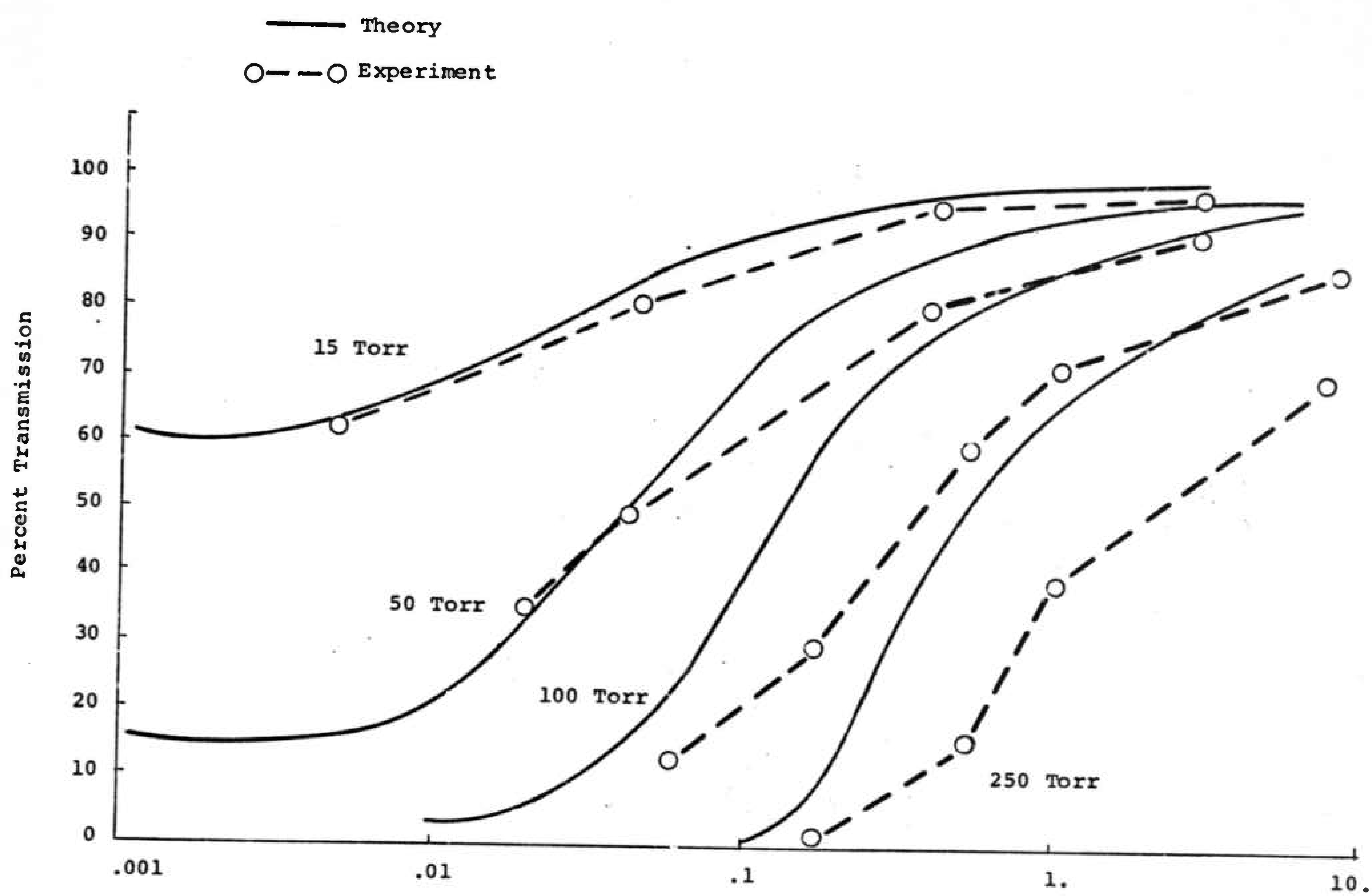

Figure 4. Single Mode SF_6 Transmission on the P(22) Line

Appendix I.

Shape and Stability Dependence of Passively Modelocked Pulses on Absorber Relaxation Time

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Abstract:

The dependence of the saturable absorber modelocked pulse shape upon the relaxation time of the absorber is determined when the laser gain is assumed constant. The pulses are found to be asymmetric with a slowly rising edge and more rapidly decaying edge. The modelocked pulsewidth is a weak function of absorber relaxation time T_A, becoming broader with increasing T_A. Furthermore, the pulses are found to experience a delay in traversing the absorber which lengthens the cavity transit time. The growth of perturbations introduced into the steady state is shown to be bounded, indicating that stable modelocked operation is possible even in cases where T_A is comparable to or greater than the width of the steady state pulse.

Shape and Stability Dependence of Passively Modelocked Pulses on Absorber Relaxation Time

P. L. Hagelstein, C. P. Ausschnitt

Introduction

Mode-locking of a laser with a passive element has been studied by many authors. Most of the theoretical studies are either computer studies[1] or qualitative analytic studies of pulse shaping, without prediction of a pulse shape[2]. One exception is the recent work by Haus[3,4], who obtains closed form expressions for the pulse shape produced by passive mode-locking in two limiting cases: (a) The relaxation time of the saturable absorber is short compared with the pulse width, (b) the relaxation time of the absorber is much longer than the pulse width.

In case (b) the laser gain medium must contribute to the pulse shaping[4,5] because, otherwise, the system would possess gain after passage of the pulse and perturbations following the pulse would experience growth; the pulse would be unstable.

In this paper we study the intermediate range of absorber relaxation time, when the absorber relaxation time is comparable to the pulsewidth. We limit our analysis to the case where the relaxation time of the gain medium greatly exceeds that of the absorber. Since there may be net gain in the system following the pulse when the absorber loss is relaxing (in the case of well-separated pulses in a system with long absorber relaxation time) the steady state pulse found may be unstable. The unstable nature of passive mode-locking with a slow absorber in the case where the gain is hard to saturate is an experimental fact[6].

Our approach is similar to that taken by Creighton et al[7], New[5], and Haus[3] in that it focusses on modifications of a pulse in one round trip transit through the laser cavity. The steady state is obtained by requiring the pulse to reproduce itself. In contrast to Creighton, the saturable absorber modulation is included self consistently. The gain dispersion, neglected by New and Letokhov in the nonlinear regime, is taken to be a major factor in the determination of pulse shape and width.

In section I we summarize the main assumptions and derive the basic equation. Section II presents some features of the pulse shape that can be gleaned directly from the basic equation. One can show that the pulse shape is in general asymmetric with a gradually rising edge. In Section III we present computer solutions which confirm this prediction and yield the two main pieces of information for the pulse as a function of T_A, its width and its net delay. Finally, in Section IV we investigate the stability of the solutions and find that, to first order, perturbation growth is bounded.

I. Assumptions and Basic Equation

The system considered here consists of three elements (see Figure 1)--the laser cavity, the active gain medium and the saturable absorber. The notation and method of derivation parallels that of Haus[3]. The following assumptions are made:

(1) The system is operating in the steady-state near threshold.

(2) The gain medium and absorber are short compared to the cavity length and the effect of each element on a pulse over a single pass is small (such that the exponential gain and loss factors can be expanded to first order). The pulses repeat with either an advance or delay from the cavity transit time, the magnitude (assumed small) and sign of which is to be determined by requiring the pulse to reproduce itself from pass to pass.

(3) As in the fast absorber mode locking solution given by Haus, we consider only the case where the pulse repetition time is much longer than the pulsewidth.

(4) The cavity loss is constant in time and the cavity modes are uniformly spaced in frequency.

(5) The active medium is homogeneously broadened and its relaxation time is much longer than the pulse repetition time (the steady state gain is approximately constant in time).

(6) The linewidth of the gain is approximated by a Lorentzian in frequency:

$$G(\omega) \quad \frac{G(\omega_o)}{1 + \frac{j(\omega - \omega_o)}{\omega_L}} \qquad (1)$$

where ω_o is the center frequency and ω_L is a measur. of the bandwidth. The pulse is assumed to have a bandwidth small compared to the linewidth so that the gain can be expanded to second order in frequency. Transforming into the time domain, the frequency dependence becomes an operator by replacing $(j\omega)^n$ by (d^n/dt^n) wherever it occurs. Thus, the gain factor operating on the pulse envelope $v(t)$ can be expressed as follows:

$$\exp\left[G\left(\frac{d}{dt}\right)\right] v \approx \left\{1 + G(\omega_o)\left[1 - \frac{1}{\omega_L}\frac{d}{dt} + \frac{1}{\omega_L^{\,2}}\frac{d}{dt^2}\right]\right\} v \qquad (2)$$

when operating on the pulse envelope.

(7) The bandwidth of the absorber is wide compared to the gain linewidth.

A two level quantum system with saturation and relaxation describes the absorber population density difference.

$$\frac{dn(t)}{dt} = -\frac{n(t) - n_e}{T_A} - \frac{\sigma_A |v(t)|^2\, n(t)}{\hbar\omega_o\, A_A} \qquad (3)$$

where $n(t)$ is the population density difference, n_e is the equilibrium density difference, T_A is the relaxation time, σ_A is the beam cross section and A_A is the beam cross section for the absorber. The degree of saturation is assumed small so that $n(t) = n_e$ in the saturation term of the rate equation. The relaxation time T_A is comparable to a pulsewidth. The loss due to the absorber is taken to be $L(t) = \sigma_A \theta_A A_A n(t)$ where θ_A is the length of the absorber.

A pulse passing through the reference plane (see Figure 1) on the n-th pass, $v_n(t)$, is modified after making a round trip pass through the cavity such that

$$v_{n+1}(t) = e^{-(\omega_o/2Q)T_R}\, e^{G(d/dt)}\, e^{-2L(t)}\, e^{G(d/dt)}\, v_n(t - T_R) \qquad (4)$$

where $\exp[-(\omega_o/2Q)T_R]$ is the constant cavity loss factor and $\exp[-2L(t)]$ is the time dependent absorber loss factor.

Requiring the pulse to be periodic in time and expanding the exponentials to first order results in the following constraint on pulse shape:

$$-\frac{\omega}{2Q} T_R \left\{ 1 + q - q\, e^{-t/T_A} \int_{-\infty}^{t} e^{t'/T_A} \frac{|v(t')|^2}{P_A} \frac{dt'}{T_A} - g \left[1 + \frac{1}{\omega_L{}^2} \frac{d^2}{dt^2} \right] + \frac{\delta + q}{\omega_L} \frac{d}{dt} \right\} v(t) = 0. \qquad (5)$$

We have defined the same quantities normalized to the cavity loss used by Haus, namely the small signal absorber loss

$$q = \frac{2\, \sigma_A\, \theta_A\, A_A\, n_e}{\frac{\omega_o}{2Q} T_R},$$

the saturated laser gain

$$g = \frac{G(\omega_o)}{\frac{\omega_o}{2Q} T_R} ,$$

the absorber saturation power

$$P_A = \frac{h\omega_o A_A}{\sigma_A T_A} ,$$

and the time shift parameter

$$\delta = \frac{\omega_L \, \delta T}{\frac{\omega_o}{2Q} T_R} ,$$

where δT is the advance of the pulse with respect to the empty cavity round trip transit.

The first term on the left of (5) is the normalized constant cavity loss. The next two terms are the normalized time-dependent absorber loss where q is a measure of the magnitude of the loss. The exponential and integral are the solution to the absorber rate equation (3). The following term is the dispersive gain of the active medium [$(g/\omega_L)(d/dt)$ acts as a delay and $(g/\omega_L{}^2 \, d^2/dt^2)$ is diffusive], where g is a measure of the magnitude of the saturated gain. The relative magnitudes of the unsaturated losses and saturated gain are such that $1 + q - g << g$. In other words, we require that the laser operate near threshold. The final term is due to pulse advance or delay, and comes about from a Taylor expansion in time:

$$v_n(t - T_p) \approx v_n(t - T_R) + \frac{d}{dt} v_n(t - T_R) \, \delta T. \tag{6}$$

The delay or advance must be determined as an eigenvalue of (5).

Equation (5) is identical to (2.6) of Haus[3], except that the response of the saturable absorber is now represented in terms of an integral. The equation can be normalized and recast in a differential form

$$\left[1 + (\delta + g)\,\frac{\omega_L \tau_p}{g}\,\frac{d}{dx} - \frac{d^2}{dx^2}\right] y - y\,\frac{T_A}{\tau_p}\,y\,\frac{d}{dx}\left\{\frac{1}{y}\left[\frac{d^2}{dx^2} - (\delta + q)\,\frac{\omega_L \tau_p}{g}\,\frac{d}{dx}\right] y\right\} - 2y^3 = 0 \qquad (7)$$

where

$$y(x) = \frac{v(t/\tau_p)}{\sqrt{\dfrac{2P_A}{q}\,(1 + q - g)}} \qquad (8)$$

$$\tau_p = \frac{1}{\omega_L}\sqrt{\frac{q}{1 + q - g}}\;. \qquad (9)$$

The time is normalized to the pulse width τ_p of the fast absorber model. τ_p is the natural time scale of the system. For a given T_A, the time shift parameter $(\delta + g)$ with respect to the cavity round-trip time as modified by the laser medium is determined as an eigenvalue. Therefore, the absorber relaxation time T_A is the only adjustable parameter in equation (7). In the limit of instantaneous relaxation, $T_A \ll \tau_p$, one obtains the fast absorber equation studied by Haus.

II. Some Consequences of the Basic Equation

We have not found a closed form solution for (5). It is possible, however, to obtain some general information on the pulse shape by considering the significance of the various terms in (5). When $T_A \to 0$, it was shown by Haus that $\delta + g = 0$, no advance or delay of the pulse takes place. This is clear from the fact that the diffusion (in time) of the pulse as caused by the dispersion preserves the symmetry of the pulse, and so does an absorber with instantaneous response. When $T_A \neq 0$, it is possible to determine the sign of $\delta + g$ by the following reasoning: an absorber with nonzero relaxation time will tend to "shave off" the front part of the pulse, hence causing a net delay, $\delta + g < 0$.

In the wings of the pulse the terms proportional to v^2 may be neglected. Hence the linear equation for the pulse shape in the wings is given by:

$$\left\{ 1 + q - g\left(1 + \frac{d^2}{{\omega_L}^2 dt^2}\right) + \frac{\delta + g}{\omega_L}\frac{d}{dt} \right\} v = 0. \qquad (10)$$

The solutions of this equation are exponentials. Hence the leading and trailing edges of the mode-locked pulses are exponentials, $\exp(st)$. The time constants of the exponentials are found to be:

$$\frac{s}{\omega_L} = \frac{\delta + g}{2g} \pm \sqrt{\left(\frac{\delta + g}{2g}\right)^2 + 1 + q - g}. \qquad (11)$$

In the front end of the pulse the growing exponential is the solution, the tail end of the pulse exhibits the decaying exponential. Since $(\delta + g)/2g$ is negative and $1 + q - g > 0$ for stability reasons, one finds for the exponential growth factor s_- in the front end.

$$s_- = \sqrt{\left(\frac{\delta + g}{2g}\right)^2 + 1 + q - g} - \left|\frac{\delta + g}{2g}\right| \tag{12}$$

The decay factor $s_+ < 0$ of the tail end is

$$|s_+| = \left|\frac{\delta + g}{2g}\right| + \sqrt{\left(\frac{\delta + g}{2g}\right)^2 + 1 + q - g} > |s_-| . \tag{13}$$

We find that the pulses must be asymmetric, exhibiting a slowly rising leading edge and more rapidly decaying edge.

III. Computer Solution

Single pulse periodic solutions to Equation (7) were found by computer. The pulses are asymmetric with the trailing wing sharper than the rising wing (see Figure 2). Asymmetric pulses of this type have been seen experimentally for a system in which the absorber relaxation time was shorter than the inverse gain linewidth [6]. The pulsewidth (full width at half intensity) τ_s is found to be only a weak function of absorber relaxation time, becoming broader with increasing T_A, as shown in Fig. 3. The eigenvalue $(\delta + g)$ as a function of absorber relaxation time is shown in Figure 4. The exponential growth and decay in the wings is correctly predicted by (12) and (13).

The absorber loss as a function of time during the passage of a pulse is shown in Figure 2 in the case of well-separated pulses and long absorber relaxation time $(T_A = 10\ \tau_p)$. The absorber loss recovers after the pulse has decayed. However, net gain exists in the system for a critical time until the absorber loss relaxes back sufficiently to cause net loss.

IV. Stability

The fact that there is net gain in the system following a pulse in the case of long absorber relaxation time may lead to instability due to perturbation growth. The evolution of a perturbation introduced into the steady state will now be considered.

The rate of change of a pulse over many passes can be written as

$$\frac{\partial}{\partial T} v(t, T) = -\frac{\omega_o}{2Q}\left\{1 + q(t) - g\left(1 + \frac{1}{\omega_L^{\,2}}\frac{\partial^2}{\partial t^2}\right) + \frac{g+\delta}{\omega_L}\frac{\partial}{\partial t}\right\} v(t, T) \tag{14}$$

where the pulse count n has been replaced by a continuous time variable, $T = n \cdot T_R$. The two time variables are assumed independent of each other; T measures the pulse repetitions, and t measures the "local" time within a pulse. In the steady state, $(\partial/\partial T)v(t, T) = 0$.

To investigate stability we introduce a perturbation δv to the steady state pulse solution v_o. The waveform in a period specified by T becomes (compare Haus' analysis in Section IV of Ref. [3])

$$v(t, T) = v_o(t) + \delta v(t, T). \tag{15}$$

The perturbation is allowed to change on the slow time scale T. In the region where the perturbation does not overlap the steady state solution its influence on the absorption and gain can be neglected. Where overlap occurs the added saturation must be included if the perturbation is in phase with the steady state. Stable operating regimes with respect to inphase perturbations have been found by Haus for the fast absorber.

In the case of the fast absorber, inphase perturbations of a stable operation decay, because the decrease of absorption caused by a perturbation is compensated by a larger decrease of gain. Quadrature perturbations are neither stabilized nor destabilized. It is reasonable to assume that, in the present case of a slow absorber, the steady state pulse is stable with respect to inphase perturbations overlapping with the pulse. On the other hand, it is clear that all perturbations following the steady state pulse and not overlapping with it grow because they do not affect the gain as produced by the steady state which is positive for some time following the pulse (we assume the saturable absorber relaxation time T_A is long enough that the net gain following the pulse persists for at least several pulsewidths). As the perturbation grows, it is pulled progressively into overlap with the steady state pulse. Perturbations in quadrature with the steady state pulse will be least stabilized because they affect neither the gain nor the loss when they come into overlap with the steady state pulse. Our objective is to determine the maximum growth possible for quadrature perturbations.

The evolution of the perturbation in T is governed by the loss modulation $q(t)$ established by the steady state pulse. In general, the perturbation experiences three major changes on the slow time scale:

(1) Amplitude growth due to the excess gain,

(2) motion toward the steady state pulse due to the fact that the steady state pulse experiences a delay in each round trip, and

(3) width variation due to the gain dispersion and loss modulation curvature.

Ultimately the perturbation moves into the steady state pulse.

If we assume a gaussian initial perturbation

$$\delta v = a_i \exp\left[-\frac{(t - t_i)^2}{2\tau_i^2}\right] \tag{16}$$

a closed form solution to (12) for the evolution of the perturbation is possible provided we divide the loss modulation following the steady state pulse into two regions as shown in Figure 5. The excess gain is approximated by a straight line in Region I, where the loss is relaxing back after being depleted by the pulse; and by a parabola in Region II, where the loss is being depleted by the steady state pulse. The results of the analysis are presented below. Details appear in Appendix A.

We neglect the change in perturbation width during its evolution. For a perturbation having an initial width approximately equal to that of the steady state pulse and much greater than the inverse laser linewidth*, i.e.

$$\tau_i \approx \tau_s >> \frac{1}{2\omega_L} , \tag{17}$$

the expression for the total amplitude growth of the perturbation initiated at $t_i = T_A/2$ assumes the form

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* Note that this is not contradictory since the steady state pulse uses only a fraction of the laser linewidth.

$$a = a_i \exp\underbrace{\left\{\left[\frac{(g+\delta)\omega_L \tau_p}{4g\sqrt{\ln 2}}\,\frac{\tau_s}{\tau_p}\right]^2\right\}}_{\text{Region II}} \exp\underbrace{\left\{\frac{\dfrac{g}{2|g+\delta|\,\omega_L\tau_p}\,\dfrac{T_A}{\tau_p}}{1+\dfrac{1}{2\ln 2}\left(\dfrac{\tau_s}{\tau_p}\right)^2 \dfrac{\tau_p}{T_A}\,\dfrac{g}{|g+\delta|\,\omega_L\tau_p}}\right\}}_{\text{Region I}} . \qquad (16)$$

The expression is separated into factors which describe the growth in the two separate regions of the loss modulation following the steady state pulse. As might be expected the growth in region I is directly dependent on the absorber relaxation time T_A, while that in region II is only dependent on T_A via the steady state "delay" $(g + \delta)$ and pulsewidth τ_s which are plotted in Figures 4 and 5.

The approximate dependence of perturbation growth on absorber relaxation time given by equation (16) is illustrated in Figure 6. Also plotted is a computer solution for a gaussian perturbation having an initial width equal to that of the steady state pulse. The difference between the computer and approximate curves for large T_A results from our neglect of shaping during the growth of the perturbation in the derivation of (16).

If, for example, we set as a criterion for stability that the perturbation amplitude at no time exceed one tenth that of the steady state pulse, then for an absorber relaxation time $T_A/\tau_p = 5$, equation (16) requires the initial perturbation amplitude a_i to be at least 40 dB down from the steady state pulse amplitude. Fundamental noise in the system, e.g. spontaneous emission noise, certainly satisfies this constraint. Other sources of noise, such as scattering from intracavity apertures, might lead to instability.

Thus, stable steady state modelocking with slow saturable absorbers is possible for some regime of $T_A > \tau_p$, without action of the laser medium as required for a truly slow saturable absorber[5] ($T_A >> \tau_p$). Stability is made possible by the fact that perturbations appearing in the system are not free to grow without bound, but are constrained by the loss modulation of the absorber to approach a steady state amplitude.

V. Conclusion

We have obtained steady state single pulse solutions for the problem of modelocking with a slow saturable absorber where the gain is assumed constant over a pulse repetition time and the pulsewidth is much less than a pulse repetition time. The pulses were found to be asymmetric with a slowly rising edge and more rapidly decaying edge. The modelocked pulsewidth is a weak function of the absorber relaxation time. In contrast to the assertions of previous authors[5] we have found that stable modelocking is possible, even in cases where the absorber relaxation time T_A is comparable to or greater than the pulsewidth. Since the steady state pulsewidth utilizes only a fraction of the laser linewidth, stable operation is possible even for $\omega_L T_A >> 1$.

Appendix A

Evolution of a Perturbation Following the Steady State Pulse

As illustrated in Figure 5 the loss modulation following the steady state pulse is divided into two regions in which the excess gain is approximated as follows:

$$\Delta g_I = \Delta g_m \left(1 - \frac{t}{t_I}\right) \qquad 0 \le t \le t_I \tag{A.1}$$

$$\Delta g_{II} = \Delta g_m \left(1 - \frac{t^2}{t_{II}{}^2}\right) \qquad -t_{II} \le t \le 0. \tag{A.2}$$

The parameters Δg_m, t_I and t_{II} characterize the magnitude and duration of excess gain respectively and are approximately related to the steady state system parameters as follows:

$$\Delta g_m = 1 + q - g \tag{A.3}$$

$$t_I = \frac{\Delta g_m}{1 + q - g + \Delta g_m} T_A = \frac{T_A}{2} \tag{A.4}$$

$$t_{II} = \frac{1}{4 \ln 2} \sqrt{\frac{\Delta g_m}{g}}\, \omega_L \tau_s{}^2 . \tag{A.5}$$

Introducing the approximation to the time dependence of the excess gain (A.1) and (A.2) into the equation of motion obeyed by the perturbation (10), we obtain

$$\frac{\partial}{\partial T}\,\delta v(t,\,T) = \frac{\omega_o}{2Q}\left[\Delta g(t) + \frac{g}{\omega_L^{\,2}}\,\frac{\partial^2}{\partial t^2} + \frac{|g+\delta|}{\omega_L}\,\frac{\partial}{\partial t}\right]\delta v. \qquad (A.6)$$

We further assume the perturbation to have a gaussian shape

$$\delta v(t,\,T) = a_o(T)\,\exp\left\{-\frac{[t - t_o(T)]^2}{2\tau_o^{\,2}(T)}\right\} \qquad (A.7)$$

characterized by a slowly varying amplitude $a_o(T)$, timing $t_o(T)$, and width $\tau_o(T)$. Initial conditions on the perturbation are taken to be

$$a_o(0) = a_i$$

$$t_o(0) = t_i$$

$$\tau_o(0) = \tau_i.$$

The use of (A.1) and (A.7) in (A.6) leads to the following set of equations governing the evolution of perturbation width, timing and amplitude in Region I.

$$\frac{d\tau_o}{dT} = \frac{\omega_o}{2Q}\,\frac{g}{\omega_L^{\,2}\tau_o} \qquad (A.8)$$

$$\frac{dt_o}{dT} = \frac{\omega_o}{2Q}\frac{|g+\delta|}{\omega_L}\left[1 + \frac{\Delta g_m\,\omega_L}{t_I{}'|g+\delta|}\tau_o^2\right] \tag{A.9}$$

$$\frac{da}{dT} = \frac{\omega_o}{2Q}\left[\Delta g_m\left(1 - \frac{\omega_o}{t_I{}'}\right) - \frac{g}{\omega_L^2\tau_o^2}\right]a. \tag{A.10}$$

If we restrict our attention to cases where

$$\frac{g}{\omega_L^2\tau_i^2}\frac{\omega_o}{Q}T << 1 \tag{A.11}$$

in Region I, then the width variation of the perturbation can be neglected and solutions for timing and amplitude are found to be

$$t_o(T) = t_i - \frac{\Delta g_m\tau_i^2}{t_I}\left(1 + \frac{t_I}{\Delta g_m\tau_i^2}\frac{|g+\delta|}{\omega_L}\right)\frac{\omega_o}{2Q}T \tag{A.12}$$

$$a_o(T) = a_i\exp\left[\Delta g_m\left(1 - \frac{t_i}{t_I}\right)\frac{\omega_o}{2Q}T + \left(\frac{\Delta g_m\tau_i}{t_I}\right)^2\left(1 + \frac{t_I}{\Delta g_m\tau_i^2}\frac{|g+\delta|}{\omega_L}\right)\left(\frac{\omega_o}{2Q}T\right)^2\right]. \tag{A.13}$$

The perturbation reaches the boundary of Region I, $t_o = 0$, at time $T = T_I$ given by

$$\frac{\omega_o}{2Q} T_I = \frac{t_i}{\frac{\Delta g_m}{t_I} \tau_i^2 \left(1 + \frac{t_I}{\Delta g_m \tau_i^2} \frac{|g + \delta|}{\omega_L}\right)} . \tag{A.14}$$

Thus, a perturbation initiated at $t_i = t_I$ has an amplitude at $t_o = 0$

$$a_I = a_i \exp\left[\frac{\frac{\Delta g_m}{|g + \delta|} \omega_L t_I}{1 + \frac{\tau_i^2}{t_I} \frac{\omega_L \Delta g_m}{|g + \delta|}}\right] . \tag{A.15}$$

The perturbation enters Region II with the initial conditions $\tau_o = \tau_i$, $a_o = a_I$. Subsequent evolution of the perturbation is governed by equation (A.6) where (A.2) is used to approximate the excess gain. To simplify the analysis we shall assume that the perturbation width on entering Region II is approximately equal to the steady state width, i. e.

$$\tau_i \approx \tau_s \tag{A.16}$$

so that changes in width can be neglected throughout. The equations governing timing and amplitude evolution then become

$$\frac{dt_o}{dT} = - \frac{\omega_o}{2Q} \frac{|g + \delta|}{\omega_L} \left(1 + \frac{t_o}{t_s}\right) \tag{A.17}$$

$$\frac{da_o}{dT} = \frac{\omega_o}{2Q} \frac{|g + \delta|^2}{4g} \left[1 - \left(\frac{t_o}{t_s}\right)^2\right] a \tag{A.18}$$

where the steady state timing approached by the perturbation is that of the steady state pulse, namely

$$t_s = - \frac{|g + \delta|}{2 \, \Delta g_m \, \omega_L} \left(\frac{t_{II}}{\tau_i}\right)^2 . \tag{A.19}$$

Equation (A.17) is of the form of a relaxation equation with relaxation rate

$$\alpha = \Delta g_m \left(\frac{\tau_i}{t_I}\right)^2 \frac{\omega_o}{Q} . \tag{A.20}$$

The slow time evolution of t_o is given by

$$t_o = t_s(1 - e^{-\alpha T}) . \tag{A.21}$$

Integrating (A.18) then gives

$$a(T) = a_I \exp\left\{ \frac{|g + \delta|^2}{4 \sqrt{\Delta g_m} \, g^{3/2}} \omega_L \, t_{II} \right\} \tag{A.22}$$

in the limit where $T \gg 1/\alpha$.

With the use of (A.3)-(A.5) equations (A.15) and (A.22) combine to give equation (14) of the text. Assumptions (A.11) and (A.16) restrict the validity of the analysis presented here to cases where the perturbation width variation can be neglected. The inclusion of width variation, however, leads to a smaller growth than predicted by (14), so that (14) represents an upper bound on the possible perturbation growth.

ACKNOWLEDGMENT

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References

[1] J. A. Fleck, "Modelocked Pulse Generation in Passively Switched Lasers," Appl. Phys., 12, 178 (1968).

[2] E. M. Gamire and A. Yariv, "Laser Modelocking with Saturable Absorbers," IEEE J. Quantum Elec., QE-3, 222 (1967).

[3] H. A. Haus, "Theory of Modelockingwith a Fast Saturable Absorber," to be published in Journal of Applied Physics.

[4] H. A. Haus, "Theory of Modelocking with a Slow Saturable Absorber," to be published in Journal of Applied Physics.

[5] G. H. C. New, "Mode-Locking of Quasi-Continuous Lasers," Optics Communications, 6, no. 6, October (1972).

[6] P. G. Kryukov and V. S. Letokhov, "Fluctuation Mechanism of Ultrashort Pulse Generation by Laser with Saturable Absorber," IEEE Journal of Quantum Elec., QE-8, no. 10, October (1972).

[7] J. R. Creighton and J. L. Jackson, "Simplified Theory of Picosecond Pulses in Lasers," J. Appl. Phys., 42, no. 9, 3409 (1971).

Figure Captions

Figure 1. Passively Modelocked Laser System

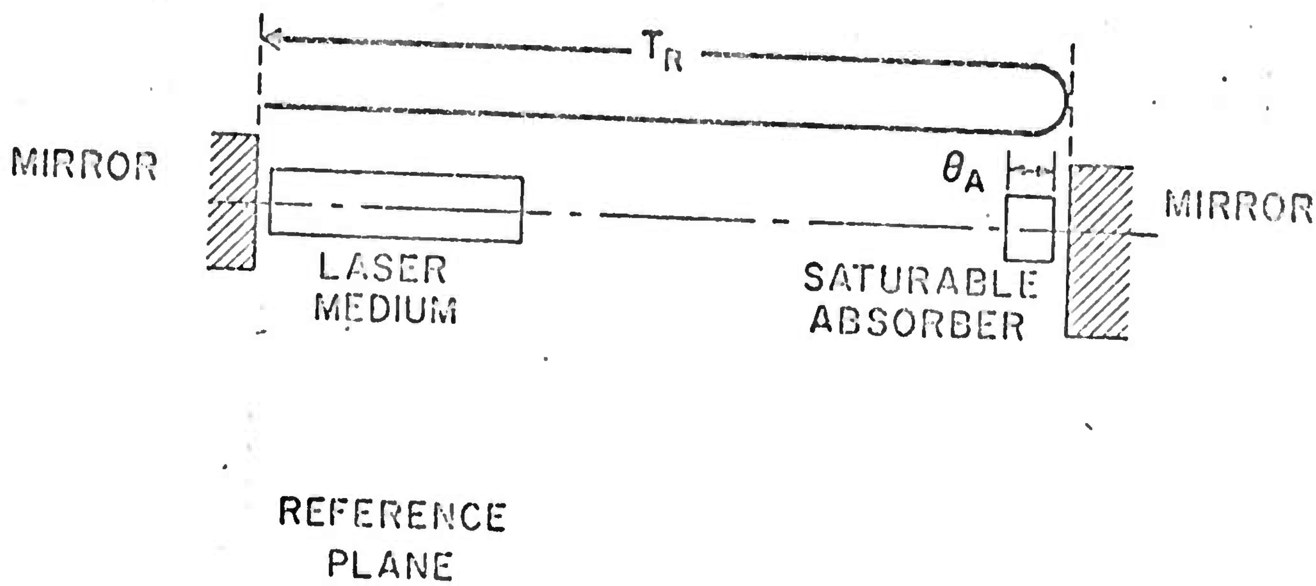
Figure 2. Absorber Loss as a Function of Time for a Well-Separated Pulse with Long Absorber Relaxation Time

Figure 3. Pulse Width Versus Absorber Relaxation Time

Figure 4. Time Shift Parameter $(\delta + g)$ Versus Absorber Relaxation Time

Figure 5. Perturbation in Region of Net Gain Following the Steady State Pulse

Figure 6. Perturbation Amplitude Growth Versus Absorber Relaxation Time



Passively Modelocked Laser System

Figure 1.

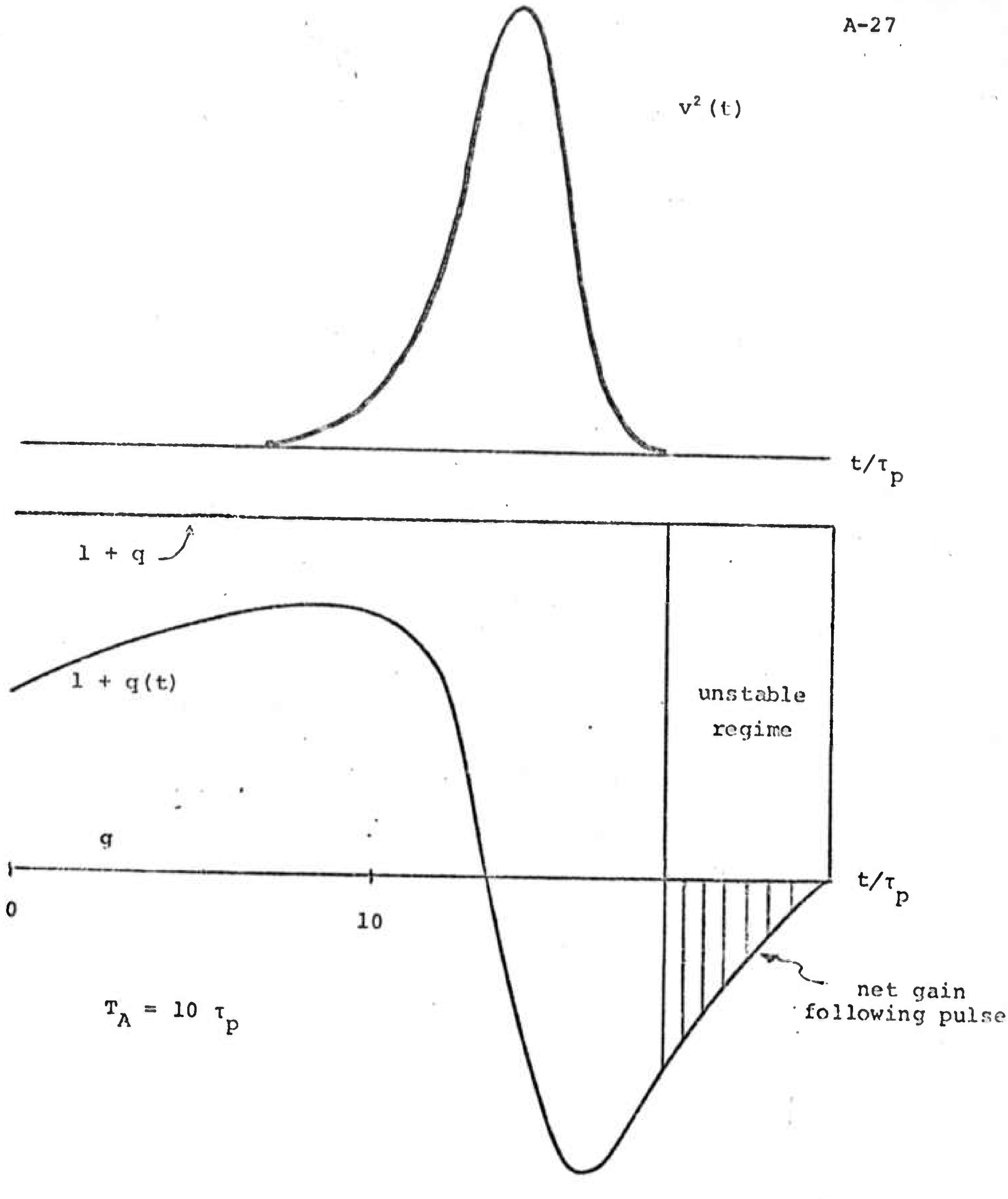


Figure 2. Absorber loss as a function of time for a well-separated pulse with long absorber relaxation time.

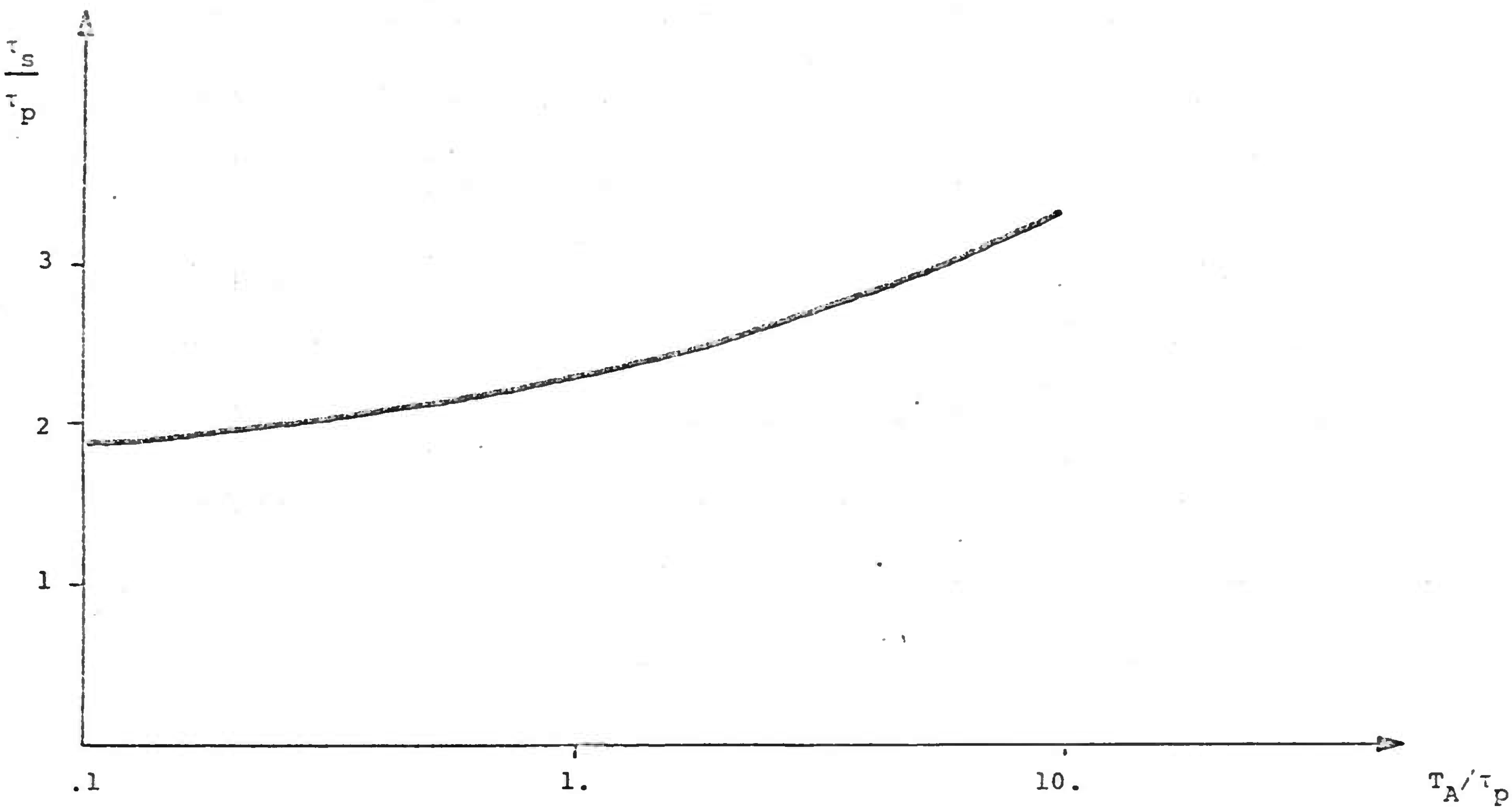


Figure 3. Pulse Width Versus Absorber Relaxation Time

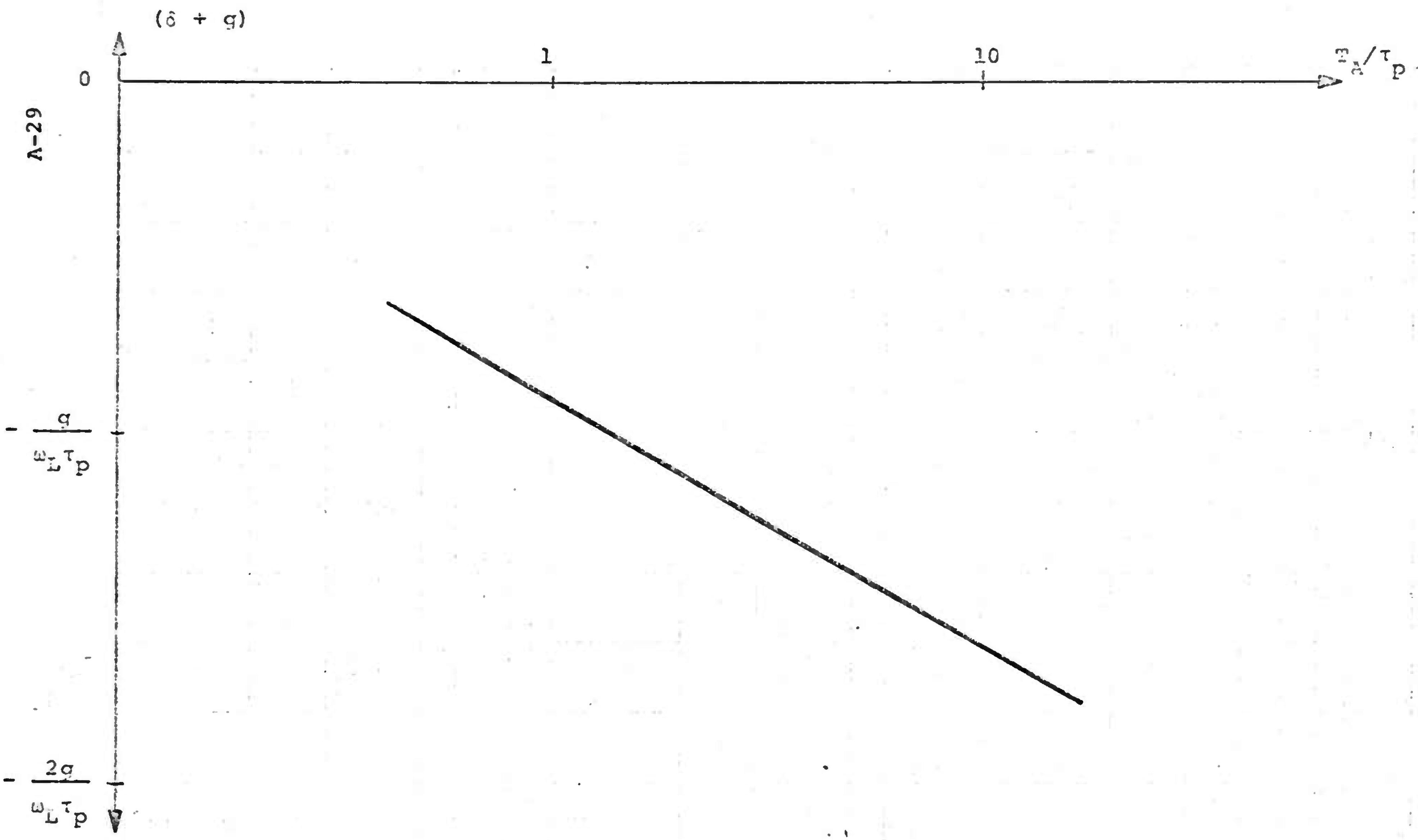


Figure 4. Time Shift Parameter $(\delta + g)$ Versus Absorber Relaxatime Time

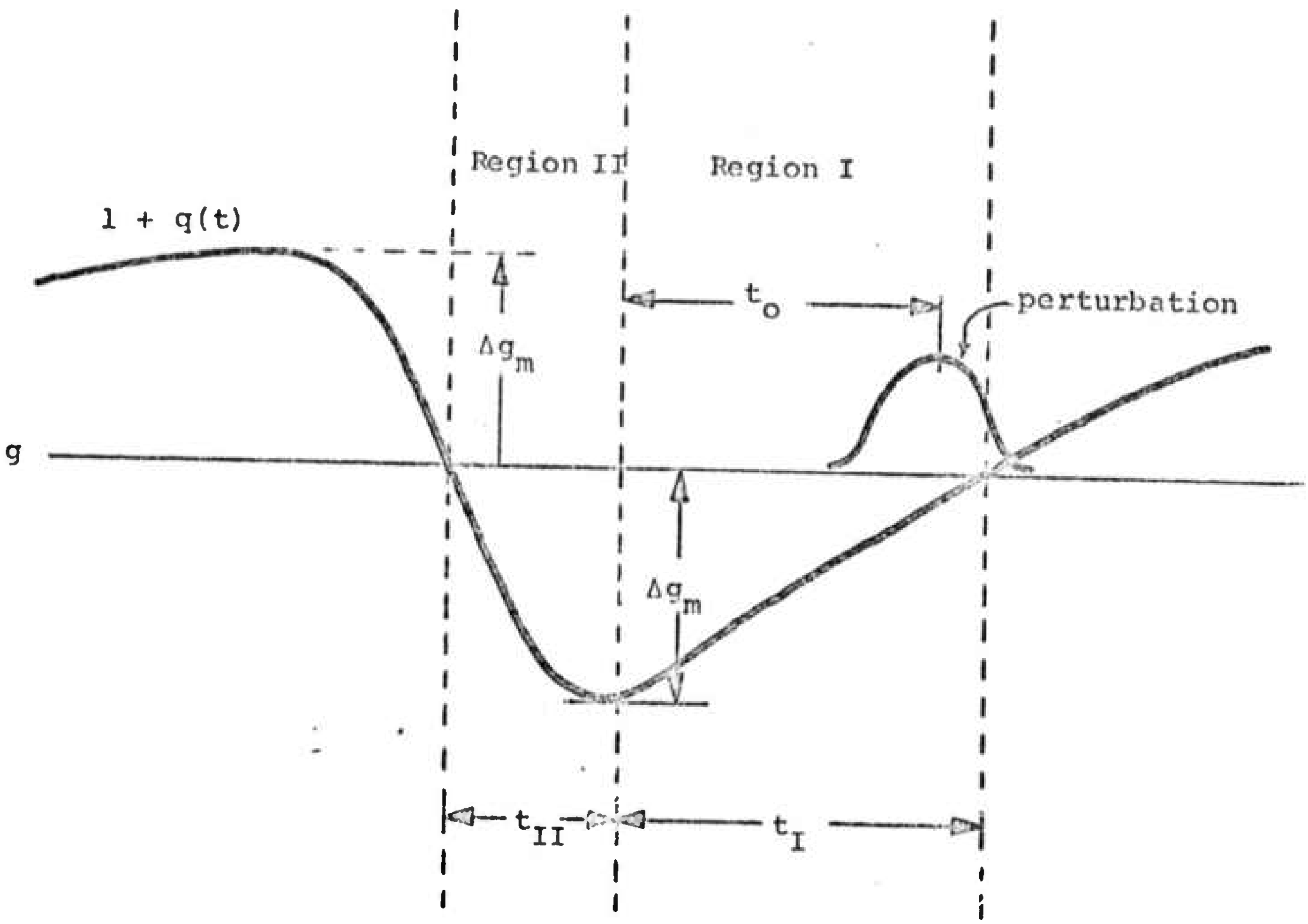


Figure 5. Perturbation in Region of Net Gain Following the Steady State Pulse

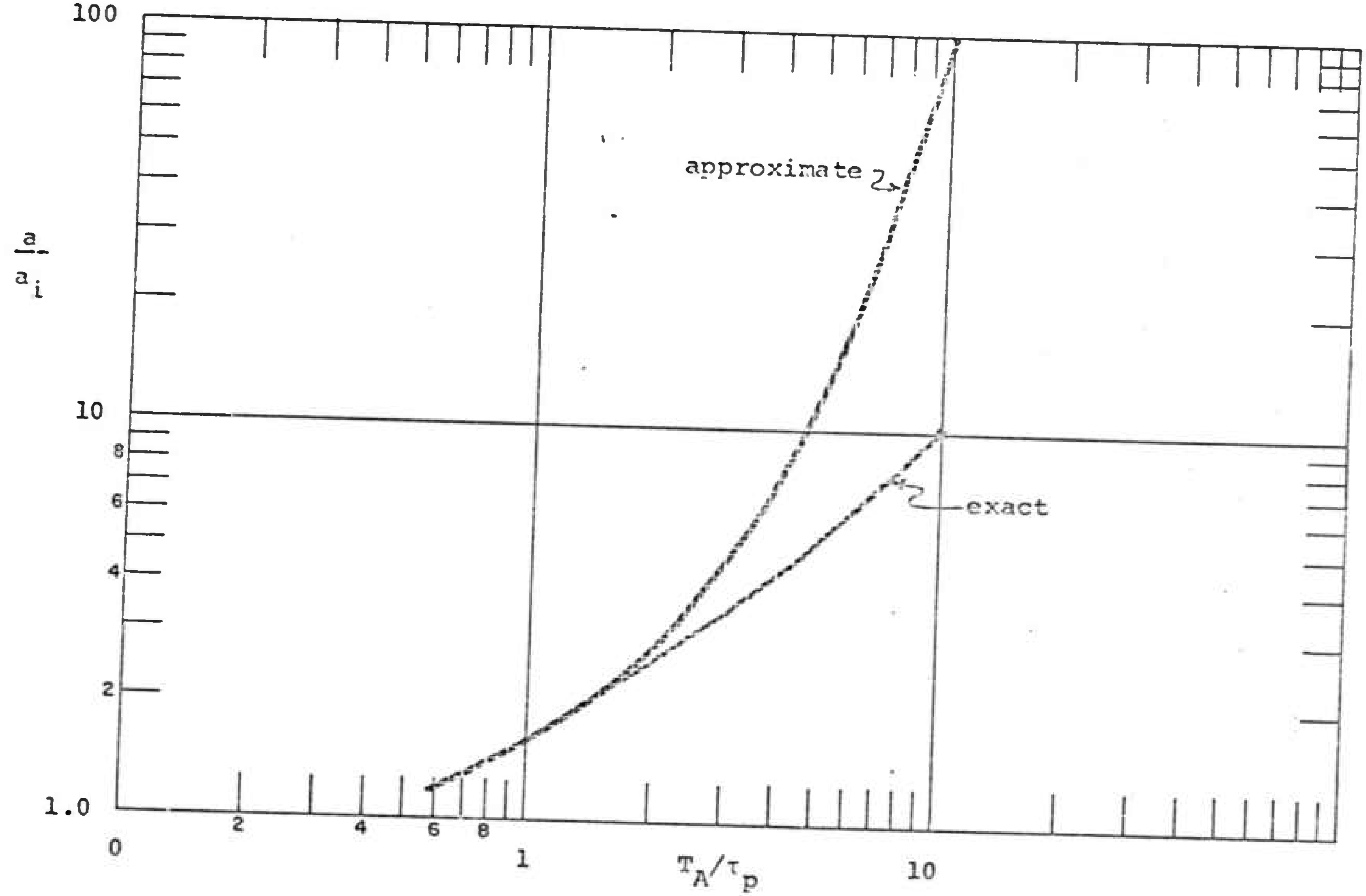


Figure 6. Perturbation Amplitude Growth Versus Absorber Relaxation Time

Appendix II.

Transient Buildup of Fast Saturable Absorber Modelocking

by

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Abstract

In the following paper we present a closed form deterministic solution to the buildup of fast saturable absorber modelocking in the homogeneously broadened laser. The evolution of the modelocked waveform from one transit through the laser cavity to the next is treated adiabatically, as a succession of quasisteady states. The secant hyperbolic pulse solution found by Haus is reached in the limit of widely separated pulses.

Introduction

Previous authors have taken a statistical approach to the study of the buildup of passive modelocking,[1-5] stating that the occurrence of modelocking relies on the selective bleaching of the absorber by the highest noise spike in a cavity transit time. Although it is true that lasing initiates with spontaneous emission noise, two facts argue against the statistical approach:

1. During the initial buildup of radiation in the cavity, where saturation of the absorber is negligible, the bandwidth of the noise is narrowed in successive passes through the cavity due to the gain dispersion.

2. Passive modelocking is consistently reproducible if the laser is operated near threshold and careful control of the system parameters is maintained.[6]

The first of the above observations indicates that the initial spontaneous emission noise is smoothed by gain-narrowing so that the sharp noise spikes are "washed out". In fact, the near threshold operation, stated in the second observation, limits the initial bandwidth of the gain medium, i.e. the number of cavity modes which see net gain. Thus, near threshold operation ensures that gain narrowing will "smooth out" the noise fluctuations. We argue, therefore, that by the time the absorber saturation becomes significant the oscillation state in the laser can be described by a perturbed single-cavity mode (SM) quasi-steady state. Consequently, the buildup of modelocking cannot be due to the selective bleaching of the absorber by a noise spike.

We show here that modelocking results from the fact that at a definite power level in the cavity the SM state to which the initial noise evolves becomes unstable with respect to a sinusoidal perturbation. Thus, an expression for the modelocking threshold, defined as the point at which the SM acquires a sinusoidal modulation is found. The subsequent evolution of the system from SM to steady state modelocked operation is deterministic and is treated as a succession of the steady state solutions to the modelocking equation derived by Haus.[7] Furthermore, for each set of system parameters--characterizing the cavity, saturable absorber and gain medium--a unique stable periodic pulse solution is reached. In the limit of well separated pulses the steady state solution is that of the single secant hyperbolic pulse found by Haus.

I. The Single Pulse Steady State Modelocking Solution

If the modelocked laser system consists of a short section of saturable absorber and a short laser gain medium, and under the assumption of weak absorber saturation, the equation governing mode-locking by a fast saturable absorber has been shown by Haus to be:[7]

$$-\frac{2}{\Delta\omega_c \, T_R} \frac{\partial v(t, n)}{\partial n} = \left[1 + q - g - \frac{q|v(t, n)|^2}{P_A} - \frac{1}{\omega_L^2} \frac{\partial^2}{\partial t^2} \right] v(t, n) \tag{1.1}$$

where

- v = field envelope
- t = time "local" to a single cavity round trip transit time
- n = integer which counts the number of round trip transits, treated as a continuum variable over many transits
- T_R = round trip transit time in cavity
- $\Delta\omega_c$ = cavity mode linewidth
- q = small signal absorber loss
- P_A = saturation power of the absorber
- g = saturated gain
- ω_L = laser medium half width

The saturated gain is related to the power in a single round trip transit by

$$g(n) = \frac{g_o(n)}{1 + \frac{P(n)}{P_L}} \qquad (1.2)$$

where $g_o(n)$ is the known small signal gain, P_L is the saturation power of the homogeneously broadened laser medium and the power is

$$P(n) = \frac{1}{T_R} \int_{-T_R/2}^{T_R/2} |v(t, n)|^2 \, dt. \qquad (1.3)$$

It should be noted that, in contrast to the fast absorber equation used by Haus, we have neglected the linewidth broadening due to gain saturation by making the substitution

$$\frac{g}{\omega_L{}^2} \simeq \frac{1 + q}{\omega_L{}^2} = \frac{1}{\underline{\omega}_L{}^2} .$$

The approximation is justified by the fact that the analysis is only valid in the regime where the gain is near threshold. Thus, gain variation is negligible compared to the normalized cavity loss, but is significant compared to $1 + q - g$. Although the following analysis can be carried out without making this approximation, the inclusion of power broadening sacrifices simplicity for little additional insight into the problem.

Also, many systems are bandwidth limited by a dispersive element (e.g. etalon) inside the laser cavity in which case the "diffusion" operator expressing pulse broadening in time is gain independent.

In the steady state the waveform v experiences no change from one transit through the cavity to the next. Thus, $\partial v/\partial n = 0$ and equation (1.1) reduces to

$$\left[1 + q - g - \frac{q|v(t)|^2}{P_A} - \frac{1}{\omega_L^2}\frac{\partial^2}{\partial t^2}\right] v(t) = 0. \qquad (1.4)$$

Haus has shown that one stable solution to (1.4) is a single pulse

$$v(t) = \frac{v_o}{\cosh \frac{t}{\tau_p}} \qquad (1.5)$$

where the pulse parameters v_o and τ_p are determined by the system parameters via the eigenvalue relations

$$1 + q - g = \frac{q}{2}\frac{v_o^2}{P_A} = \frac{1}{\omega_L^2 \tau_p^2} \qquad (1.6)$$

in conjunction with (1.2). In the following section we show that the general solution to (1.4) is periodic, and that (1.5) holds only in the special case where $T_R >> \tau_p$.

II. The Periodic Steady State Modelocking Solution

The secant hyperbolic steady state solution found by Haus is only one solution to equation (1.4). The fact that it is a single pulse or "soliton" solution implies that it corresponds to the case where the cavity transit time is much longer than the pulsewidth, $T_R >> \tau_p$, so that the inherent periodicity of the system can be ignored. During the buildup of modelocking from SM, however, it is clear that the periodicity of the system must be taken into account since the pulsewidth τ_p is initially comparable to T_R. As a first step toward understanding the transient buildup, therefore, we must find the periodic solution to (1.4).

In general, equation (1.4) is the equation of motion of a particle in the potential well

$$U(v) = -\frac{qv^4}{4P_A} + \frac{1}{2}(1 + q - g)\, v^2 + C \qquad (2.1)$$

where C is a constant. If the particle is launched with zero velocity at the well height 0, corresponding to a displacement v_o, it oscillates between the turning points defined by the roots of $U(v)$. Thus, the solution is periodic, with time dependence

$$v(t) = v_o \sqrt{1 - \gamma\, sn^2\left(\frac{t}{\tau_p}, \gamma\right)} \qquad (2.2)$$

where sn is a Jacobian elliptic function of t/τ_p of modulus γ[8] and, for convenience, we have defined the new constant γ as

$$\gamma = \frac{q\, v_o^4}{4P_A} C + 1. \tag{2.3}$$

Substitution of (2.2) in (1.4) gives the eigenvalue relations

$$\frac{q\, v_o^2}{2P_A} = \frac{1}{\underline{\omega}_L^2 \tau_p^2} \tag{2.4}$$

and

$$1 + q - g = \frac{1}{\underline{\omega}_L^2 \tau_p^2} (2 - \gamma) \tag{2.5}$$

which determine the peak amplitude v_o and "pulsewidth" τ_p within each period. The periodicity of the solution, which we require to equal the cavity transit time T_R, determines the constant γ via the relation

$$2K(\gamma) = \frac{T_R}{\tau_p} \tag{2.6}$$

where K is the complete elliptic integral of the first kind of argument γ. In the limit $T_R >> \tau_p$, equation (2.6) dictates that $\gamma \to 1$ and equations (2.2 - 2.4) become the secant hyperbolic solution and eigenvalue relations found by Haus (1.5 - 1.6). Requiring the roots of (2.1) to be real sets the limits on γ,

$$0 < \gamma < 1 \tag{2.7}$$

In turn, (2.7) sets limits on the relative magnitude of T_R and τ_p via (2.6), namely

$$\pi < \frac{T_R}{\tau_p} < \infty. \tag{2.8}$$

The lower limit of (2.8) corresponds to the case where the wave-form v is SM with an infinitesimal sinusoidal ripple one half wavelength of which is fitted in a round trip transit. As noted earlier the upper limit is the case where v is a single secant hyperbolic pulse.

The power in a single transit time is determined by substituting the general solution (2.2) in equation (1.3). The integral of $|v|^2$ over one period is expressible in terms of E, the complete elliptic integral of the second kind of argument γ, so that (1.3) gives

$$P = \frac{2v_o{}^2\tau_p}{T_R} E(\gamma) = v_o{}^2 \frac{E(\gamma)}{K(\gamma)} . \tag{2.9}$$

III. The Modelocking Threshold

Lasing initiates when the small signal gain g_o first breaks above the small signal loss $1 + q$. In the region near the onset of lasing we assume that the saturation of the absorber is negligible. The initial buildup of the field is described by

$$\frac{2}{\Delta\omega_c T_R} \frac{\partial v(n, t)}{\partial n} = \left[g - \left(1 + q\right) + \frac{1}{\omega_L^2} \frac{\partial^2}{\partial t^2} \right] v(n, t) . \tag{3.1}$$

We can express the field in the nth transit by a superposition of the cavity modes, namely

$$v(n, t) = \sum_k V_k \exp \left[j \frac{k 2\pi t}{T_R} + \int^n \beta_k(n)\, dn \right] \tag{3.2}$$

where k is an integer which labels the modes, V_k is the amplitude and β_k is the gain coefficient characteristic of the kth mode.

Substitution of (3.2) in (3.1) gives an expression for the gain coefficient

$$\frac{2}{\Delta\omega_c T_R} \beta_k(n) = g(n) - (1 + q) - \frac{1}{\omega_L^2} \left(k \frac{\pi}{T_R} \right)^2 . \tag{3.3}$$

The mode nearest line center $(k = 0)$ experiences the most rapid growth due to the band narrowing effect of the gain dispersion. Thus, operation on the single mode (SM) at line center will dominate. We assume, therefore, that the laser prior to the onset of modelocking is in SM operation, namely that its output does not vary over the transit time T_R.

Thus far we have neglected the saturation of the absorber. Ultimately, however, the SM described by $v_o(n)$ grows to the point where the equations describing the subsequent growth of the field must include the absorber saturation. Hence, the equation governing growth becomes

$$\frac{dv_o}{dn} = \beta_o(n)\; v_o(n) \tag{3.4}$$

where β_o is given by

$$\frac{2\beta_o(n)}{\Delta\omega_c\, T_R} = g(n) - (1 + q) + \frac{q\; v_o{}^2(n)}{P_A} \; . \tag{3.5}$$

Note that gain dispersion plays no role in determining β_o, the gain coefficient of the mode at line center. Since the field is "time independent" within T_R the saturated gain is

$$g(n) = \frac{g_o(n)}{1 + \dfrac{v_o{}^2(n)}{P_L}} \; . \tag{3.6}$$

The modelocking threshold is determined by the value of $v_o(n)$ for which the first modelocking solution to (1.4) is obtained. As shown by (2.8), the first solution is the case where the SM waveform acquires a sinusoidal ripple, with one half wavelength fitted in a round trip transit. Thus, the modelocking threshold is defined by $n = \ell$ such that

$$\frac{T_R}{\tau_p(\ell)} = \omega_L T_R \sqrt{\frac{q}{2P_A}}\, v_o(\ell) = \pi \tag{3.7}$$

and we have

$$v_o(\ell) = \frac{\pi}{\omega_L T_R} \sqrt{\frac{2P_A}{q}} \, . \tag{3.8}$$

The SM field must build up to a value $v_o(\ell)$ before modelocking initiates.

To prove that the transition from the SM to the first modelocking solution is deterministic, we now show that a sinusoidal perturbation on the SM experiences more rapid growth than the SM. In other words, the system must modelock rather than continue to run in a single mode.

Returning to equation (1.1), we assume a perturbed solution of the form

$$v(n, t) = v_o(n) + \sum_{k \neq 0} \delta v_k \exp\left[jk \frac{2\pi t}{T_R} + \int^n \beta_k(n)\, dn\right] \tag{3.9}$$

where $v_o(n)$ is the growing SM solution whose gain coefficient β_o is given by (3.5). The perturbation has been expanded in the cavity modes. Substituting (3.9) in (1.1) we find that to first order $\beta_k(n)$ is given by

$$\frac{2}{\Delta\omega_c T_R}\beta_k(n) = g(n) - (1 + q) + \frac{3q}{P_A} v_o^2(n) - \frac{1}{\omega_L^2}\left(k\,\frac{2\pi}{T_R}\right)^2 . \tag{3.10}$$

Since the perturbation is "orthogonal" to the single mode field over a cavity transit time, the saturated gain does not change to first order in the perturbation amplitude, and is still specified by (3.6). From (3.5) we see that the difference between the gain coefficient of the single mode field and that of the perturbation is

$$\frac{2}{\Delta\omega_c T_R}(\beta_k - \beta_o) = \frac{2q}{P_A} v_o^2 - \frac{1}{\omega_L^2}\left(k\,\frac{2\pi}{T_R}\right)^2 . \tag{3.11}$$

Maximum growth is experienced by the perturbation for which $k = 1$. Setting $\beta_1 = \beta_o$ then extablishes the value of v_o for which the perturbation growth first exceeds the SM growth. Clearly the threshold value of v_o predicted by (3.11) is the same as that given by (3.8). Thus, for $v_o > v_o(\ell)$ the sinusoidal perturbation grows faster than the SM. Consequently, once the threshold $v_o = v_o(\ell)$ is reached, the system must ease into the modelocking solution.

IV. The Transient Buildup Solution

We have shown in the previous section that at a particular SM field amplitude in the cavity the single mode field must break into oscillation. Furthermore, the initial oscillation corresponds to the first steady state solution of the modelocking equation. The transient evolution from the initial oscillation to the final steady state is now treated as succession of quasi-steady state solutions to equation (1.1), where the left hand side of the equation dictates a change in the waveform $v(n, t)$ from one transit to the next.

To determine precisely how the waveform evolves we constrain the buildup to be adiabatic, namely the increase in the field energy in one transit is set equal to the energy supplied by the system in one transit so that the total energy remains constant. The energy balance equation

$$\Delta W_{field} = - \Delta W_{system} \tag{4.1}$$

can be obtained directly from (1.4) by multiplying both sides by $v^*(n, t)$ and integrating over one period or transit time. Thus, (4.1) becomes

$$\frac{d}{dn} \oint |v|^2 \, dt = \Delta\omega_c T_R \left\{ [g - (1 + q)] \oint |v|^2 \, dt + \frac{q}{P_A} \oint |v|^4 \, dt - \frac{1}{\omega_L^2} \oint \left|\frac{dv}{dt}\right|^2 dt \right\} \tag{4.2}$$

where $\oint$ denotes integration over one period $-T_R/2 < t < T_R/2$.

We assume that the power in the field experiences a charge over one transit time given by

$$P(n+1) = [1 + 2\alpha(n)]\, P(n) \tag{4.3}$$

where $\alpha(n)$ is the growth rate of the field to be determined from the energy balance equation. Thus, for two energy states which are differentially close, the left hand side of the energy balance equation becomes simply

$$\Delta W_{field} = \frac{d}{dn} \oint |v|^2 dt = 2T_R\, \alpha(n)\, P(n) . \tag{4.4}$$

In order to evaluate the change in the system energy we make use of the fact that the quasi-steady state solution for each value of n is given by (2.2), namely

$$v^2(n,\ t) = \frac{K[\gamma(n)]}{E[\gamma(n)]}\, P(n) \left\{ 1 - \gamma(n)\ \mathrm{sn}^2 \left[\frac{t}{\tau_p(n)}\ ,\ \gamma(n) \right] \right\} \tag{4.5}$$

where

$$\tau_p^{\,2}(n) = \frac{1}{\omega_L^{\,2}}\, \frac{2\, P_A}{q\, P_L}\, \frac{E[\gamma(n)]}{K[\gamma(n)]}\, \frac{P_L}{P(n)} . \tag{4.6}$$

The constant $\gamma(n)$ and the saturated gain $g(n)$ are determined by the relations

$$2K[\gamma(n)] = \frac{T_R}{\tau_p(n)} \tag{4.7}$$

and

$$g(n) = \frac{g_o(n)}{1 + \frac{P(n)}{P_L}} . \tag{4.8}$$

Substitution of (4.5) in the right hand side of (4.2) and equating to (4.4) gives the desired expression for $\alpha(n)$

$$\frac{2}{\Delta\omega_c T_R} \alpha(n) = [g(n) - (1 + q)] + [2 - \gamma(n)] \frac{K[\gamma(n)]}{2E[\gamma(n)]} \frac{P(n)}{P_A} . \tag{4.9}$$

The equations governing the transient buildup are now complete. The waveform $v(n, t)$ for each value of n is given by (4.5) where the pulsewidth in each period $\tau_p(n)$, the constant $\gamma(n)$ and the saturated gain $g(n)$ are determined by equations (4.6 - 4.8). The evolution of the nth waveform to the $(n + 1)^{th}$ waveform is dictated by equation (4.3), where conservation of energy constrains $\alpha(n)$ to be given by (4.9).

The initial condition on $P(n)$ was shown in section III to be

$$P(m) = \frac{\pi^2}{\omega_L T_R} \frac{2P_A}{q} \quad . \tag{4.10}$$

The final steady state defined by

$$v(t, n + 1) = v(t, n), \tag{4.11}$$

requires that

$$\frac{2\alpha(n)}{\Delta\omega_c T_R} = g(n) - (1 + q) + \frac{q\; P(n)\; E[\gamma(n)]}{2P_A\; K[\gamma(n)]}\; [2 - \gamma(n)] = 0. \tag{4.12}$$

We recognize that (4.12) is the eigenvalue relation (2.5) determined earlier to describe the general steady state solution. Equations (4.12) and (4.6 - 4.8) now uniquely specify the final steady state solution as was shown in section II.

Of course, in cases where the small signal gain of the laser medium never reaches a constant value (e.g. gain or Q switched lasers) condition (4.11) is never satisfied. This does not limit the effectiveness of the analysis provided the gain variation over the cavity transit time T_R is small. In fact, the analysis enables us to trace the complete evolution of transient modelocking from start to finish.

References

[1] V. S. Letokhov, Soviet Phys. JETP, 28, no. 3, March 1969, p. 562.

[2] P. G. Kryokov, V. S. Letokhov, IEEE J. Quantum Electron., QE-8, no. 10, Oct. 1972.

[3] T. I. Kuznetsova, Sov. Phys. JETP, 30, no. 5, May 1970, p. 904.

[4] R. Wilbrandt, H. Weber, IEEE J. Quantum Electron., QE-11, no. 5, May 1975, p. 186.

[5] B. Hausherr, E. Mathieu, H. Weber, IEEE J. Quantum Electron., QE-9, no. 4, April 1973, p. 445.

[6] R. Fortin, F. Rheault, J. Gilbert, M. Blanchard, J. L. Lachambre, Can. J. Phys., 51, 1973, p. 414.

[7] H. A. Haus, J. Appl. Phys., 46, no. 7, July 1975, p. 3049.

[8] E. T. Whittaker, G. N. Watson, A Course of Modern Analysis, University Press, Cambridge, 1969, p. 492.